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A THEORY OF COMPRESSIBLE VISCOUS FLOW WITH APPLICATIONS TO LATE-TIME FIREBALL MIXING

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A THEORY OF COMPRESSIBLE VISCOUS FLOW WITH APPLICATIONS TO LATE-TIME FIREBALL MIXING

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PREFACE

This report discusses the motion of gases and weapon debris and the temperature distribution in the immediate vicinity of the rising late-time fireball of a nuclear detonation in the atmosphere. "Late-time" loosely denotes that period after the fireball has adjusted to the pressure of the ambient atmosphere. This subject will be of interest to those agencies which seek methods for determining the influence of the fireball on electromagnetic wave propagation.

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SUMMARY

This report describes a technique for the solution of a class of problems in compressible turbulent flow and demonstrates its application to a problem involving the rise and the entrainment of ambient air by the late-time fireball due to a nuclear detonation in the atmosphere. The method is based on the work of J. E. Moyal[1] which shows that the components of Fourier spectra of velocity parallel to the wave vector are associated with compression waves (irrotational flow), while those components transverse to the wave vector are associated with eddy turbulence. If solutions are sought for systems in pressure equilibrium with an undisturbed atmosphere, the resulting formalism takes on considerable simplicity due to uncoupling of the equations of motion.

Assumptions related to the evaluation of certain convolution integrals which appear in the theory are examined and shown to involve the largest eddies in the role of the principal convective agent. This assumption converts the nonlinear Navier-Stokes equation to linear form with solutions conveniently described in a moving frame of reference. The heat transfer and continuity equations are also simplified by the same linearization process. Simultaneous solutions of the equations in wave-vector space and subsequent inversion to physical space with certain initial conditions provides solutions which have considerable similarity to the actual flow situation fround in weapons tests such as vortex ring (torus) formation. Qualitative and quantitative features of the predicted motion, for example, velocity distribution, rise rate, and temperature, are discussed.

A section on conclusions and recommendations assembles the main ideas and results which have come from the study and suggests the direction of future effort both with respect to the fireball problem and with respect to the general theory of turbulent flow.

A set of appendices is supplied which provides information on the literature in the field of turbulent flow, a method of solving the linearized equations of Moyal for large Reynolds number, and provides information on certain integrals used in the text.

TABLE OF CONTENTS

LIST OF SYMBOLS

a	velocity of sound
b	initial condition index
g	acceleration of gravity
γ	ratio of specific heats
H	scale height in the atmosphere
i	$= \sqrt{-1}$
k	Fourier mode index, modulus of wave vector
$\varkappa$	thermal diffusivity
ν	kinematic viscosity
p	pressure
Pr	Prandtl number
π_p	pressure function ($= \ln p/p_o$)
r	radial coordinate in spherical system or typical length
Re	Reynold's number
ρ	density or radial coordinate in cylindrical system
dQ	Fourier spectra of temperature function
dS	Fourier spectra of density
s	density function ($= \ln \rho/\rho_o$)
T	absolute temperature
t	time
ϑ	temperature function ($= \ln T/T_o$)
v_i	i th component of velocity
x_i	space coordinate in rectangular system
dZ_i	Fourier spectra of i—component of velocity.

Subscripts

k denotes component of the spectra of velocity along the wave vector k

o denotes reference conditions far from the fireball, initial conditions or denotes typical values of velocity and length in the flow field

oc denotes conditions at fireball center at $t = 0$

i denotes vector component ($i = 1, 2, 3$).

Superscripts

prime denotes dimensional variable

(t) denotes part of velocity spectra transverse to wave vector.

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SECTION 1

INTRODUCTION

A task on "Investigation of the Effects of Nuclear Detonations on Electromagnetic Propagation," Contract AF 19(628)-4210, has been the study of turbulent mixing processes. The objective of this study has been to obtain information on the mixing processes in the fireball which might determine the distribution of debris, temperature, and velocity.

The motion and temperature distribution of the late-time fireball has considerable importance from the point of view of the weapons system engineer. This is related to at least two separate areas of penetration aid and ABM system design: the absorption of EM radiation (such as ECM and radar waves) which is dependent on temperature distribution and debris motion; and clutter which is due to irregularities of the properties of the fluid. Both of these effects strongly influence electronic system design. Because of this importance a number of investigations of the phenomenon associated with the rise and entrainment of the fireball have been conducted.* Test information has been correlated and empirical expressions developed for average temperature, rise velocity, and fireball radius.[4] These expressions are useful kinds of information for the system designer since over certain ranges of yield and altitude of detonation they provide data for the gross features of the fireball such as an average temperature. The variations of temperature and the detailed motions in the vicinity of the rising fireball have not been forthcoming up to now largely because the problem involves the complicated concepts of gravitational convection in a compressible, viscous, heat conducting fluid.

This report attempts to provide some further information on the fireball rise problem by using the most general methods of fluid mechanics. The problem is first posed in as complete a form as possible. Throughout the analysis a series of assumptions is invoked to continuously simplify the problem in a way which preserves the main mechanisms of the dynamics and still allows solutions to be obtained which display some important features of the problem.

*See References 2, 3, 4 and 5.

Unfortunately the problem in its fullest extent must be posed within the framework of compressible turbulent flow. There has been only a small amount of fundamental work* (see Appendix B) in this field although applications involving mixing length or eddy viscosity assumptions in wakes, jets, and supersonic boundary layers are highly developed. The work on turbulent gravitational convection has involved assumptions such as similarity[10] or restrictions on the role of density differences.[11]

Although much important insight into the problem has been made available, by this means it has been difficult to find full solutions which yield the kinds of motion which experience shows the fireball to undergo such as the formation of a moving vortex ring. Because of this it appeared that a different class of assumptions would lead to more realistic motions.

O. M. Phillips[12] has shown that if the net linear momentum of the fluid is non-zero in a viscuous fluid in the final stages of decay of turbulence, the motion develops into a kind of vortex ring similar to the solutions found in this report. The basic difference is that the inclusion of part of the non-linear term as carried out here provides a translation of the ring through the fluid which depends on the initial conditions. Phillips demonstrates that the low density in the core of a heated vortex ring may be expected to keep the material of the ring coherent, presumably because of the effect of centrifugal forces operating on the density differences. The theory of vortex rings in an inviscid fluid due to Lamb and to Hill are reviewed by Turner.[13, 14] Some excellent photographs of rings are contained in Reference 14.

In an important paper Moyal[1] demonstrates the methodology by which two types of velocity spectra may be segregated. The resulting equations are tractable if the assumption is made that the largest eddies in the flow (the only ones which couple with the constant gravitational field) dominate the non-linear terms. This last assumption appears reasonable when the equations of the velocity spectra are examined.

The convolution integrals which arise from the non-linear terms after Fourier transformation are approximated by the value of the integrand which corresponds to non-linear coupling between the largest eddies and the eddies of all other sizes. It is shown that the resulting term is responsible for translation of the space coordinates in a manner which depends on the initial conditions. The translation effect allows

*See References 6, 7, 8 and 9.

a convenient description of the motion and temperature distribution in a moving frame of reference. For the case of an initial velocity in the upward direction only, the translational velocity may be identified with the rise velocity of the fireball.

SECTION 2

THE EQUATIONS OF MOTION, HEAT TRANSFER, AND STATE IN PHYSICAL SPACE

The problem is specified in terms of the conservation of mass, momentum, energy, and the equation of state using the dependent variables s, π_p, and θ where $s = \ln(\rho/\rho_o)$, $\pi_p = \ln(p/p_o)$, $\theta = \ln(T/T_o)$ and the velocity, $\vec{v} = (v_1, v_2, v_3)$ of the fluid. The subscript zero corresponds to any convenient time-independent reference condition and ρ, p, and T are the density, pressure and temperature at any point $\vec{x}' = (x_1', x_2', x_3')$ and any time t'. We specify a body force per unit mass of g in the negative x_3' direction.

$$\frac{\partial s}{\partial t'} + \frac{\partial v_j}{\partial x_j'} = -v_j \frac{\partial s}{\partial x_j'} \quad . \tag{1}$$

$$\frac{\partial v_i}{\partial t'} - \nu \nabla^2 v_i - \frac{1}{3} \nu \frac{\partial}{\partial x_i'} \left(\frac{\partial v_j}{\partial x_j'} \right) + \frac{a'^2}{\gamma} \frac{\partial \pi_p}{\partial x_i'} = -v_j \frac{\partial v_i}{\partial x_j'} - g\,\delta_{i3} \quad . \tag{2}$$

$$\frac{\partial \theta}{\partial t'} - \varkappa \nabla^2 \theta - \frac{(\gamma-1)}{\gamma} \frac{\partial \pi_p}{\partial t'} = -v_j \frac{\partial \theta}{\partial x_j'} + \frac{(\gamma-1)}{\gamma} v_j \frac{\partial \pi_p}{\partial x_j'} + \frac{\nu}{c_p T}$$
$$\left[\frac{1}{2} \left(\frac{\partial v_i}{\partial x_j'} + \frac{\partial v_j}{\partial x_i'} \right)^2 - \frac{2}{3} \left(\frac{\partial v_j}{\partial x_j'} \right)^2 \right] \quad . \tag{3}$$

$$\pi_p = s + \theta \quad . \tag{4}$$

The summation convention is used in these equations and throughout this report. In the order given, the equations represent the conservation of mass, momentum, and energy and the equation of state. The quantities ν, $\varkappa$ are the kinematic viscosity and thermal diffusivity of the fluid, respectively. The quantities a', γ, c_p are the

velocity of sound, ratio of specific heats, and the specific heat at constant pressure of the fluid, respectively. The quantity δ_{i3} is equal to unity if $i = 3$ and zero otherwise. In performing the following analysis we make the assumption that the quantities ν, $\varkappa$, a', γ, and c_p are constant.

In many problems involving the late-time fireball, with low Mach number rise rate, the term in braces on the right-hand side of Equation 3, representing the effect of viscous dissipation on the temperature, is small compared to the other terms and may be disregarded.[1] This does not imply that viscous dissipation is neglected insofar as it is the ultimate energy sink in the smallest eddies. It is simply not considered to influence the temperature measurably.

In addition, the equation of state represents a perfect gas and we, therefore, assume that the effect of chemical reactions within the fluid plays a negligible role. This contributes to inaccuracy of the solutions when the temperature is above, say, 2000°K at most altitudes.

It is implicitly assumed in writing the above equations that the flow behaves according to continuum mechanics.

We have also assumed that the fireball is small compared to distances over which the earth's gravitational field changes by a significant amount.

The late-time fireball is characterized by a relatively uniform pressure at a given altitude. This concept of "pressure equilibrium," though it is probably not completely valid, will be employed to simplify the equations. This assumption requires that the pressure everywhere in the fireball is equal to that which occurs in an undisturbed atmosphere. This requires that

$$\frac{a'^2}{\gamma}\frac{\partial \pi_p}{\partial x_i'} = -g\,\delta_{i3} \quad , \tag{5}$$

$$\frac{\partial \pi_p}{\partial x_3'} = -\frac{g\gamma}{a'^2} \; ; \quad \frac{\partial \pi_p}{\partial x_1'} = \frac{\partial \pi_p}{\partial x_2'} = \frac{\partial \pi_p}{\partial t'} = 0 \quad , \tag{6}$$

$$\pi_p = -(g\gamma/a'^2)\, x_3' + \pi_p\Big|_{x_3'=0} = -x_3'/H' + \pi_p\Big|_{x_3'=0} \quad , \tag{7}$$

where H' is the scale height of pressure in the atmosphere and is given by

$$H' = \frac{a'^2}{g\gamma} \quad ,$$

and

$$\frac{\partial \pi_p}{\partial x'_3} = -1/H' \quad .$$

If Equations 5 and 6 are employed in the momentum and energy equation these become

$$\frac{\partial v_i}{\partial t'} - \nu \nabla^2 v_i - \frac{1}{3} \nu \frac{\partial}{\partial x'_i} \frac{\partial v_j}{\partial x'_j} = - v_j \frac{\partial v_i}{\partial x'_j} \tag{8}$$

$$\frac{\partial \theta}{\partial t'} - \varkappa \nabla^2 \theta = - v_j \frac{\partial \theta}{\partial x'_j} - \frac{\gamma - 1}{\gamma} v_3/H' \quad . \tag{9}$$

The other equations remain unchanged. Equations 1, 4, 8, and 9 are the basic set which will be discussed in the following sections.

SECTION 3

THE NON-DIMENSIONAL EQUATIONS IN WAVE-VECTOR SPACE

Following the work of Moyal the set of Equations 1, 4, 8 and 9 is transformed into wave-vector space in accordance with the following relations

$$dZ_i = \frac{d\vec{k}}{(2\pi)^3} \int v_i \, e^{-i\vec{k}\cdot\vec{x}'} \, d\vec{x}' \; ; \quad v_i = \int e^{+i\vec{k}\cdot\vec{x}'} \, dZ_i \quad . \qquad (10)$$

The quantity dZ_i is the Fourier spectra of the velocity in the i direction. The integral on the right is shown in Stieltjes form to allow for the possibility that dZ_i may be a "pathological" function (or improper function in the sense of the Dirac delta function).

Similarly, we define formally the quantities dS, dP, and dQ as follows:

$$dS = \frac{d\vec{k}}{(2\pi)^3} \int s \, e^{-i\vec{k}\cdot\vec{x}'} \, d\vec{x}' \; ; \quad s = \int e^{+i\vec{k}\cdot\vec{x}'} \, dS$$

$$dP = \frac{d\vec{k}}{(2\pi)^3} \int \pi_p \, e^{-i\vec{k}\cdot\vec{x}'} \, d\vec{x}' \; ; \quad \pi_p = \int e^{+i\vec{k}\cdot\vec{x}'} \, dP$$

$$dQ = \frac{d\vec{k}}{(2\pi)^3} \int \theta \, e^{-i\vec{k}\cdot\vec{x}'} \, d\vec{x}' \; ; \quad \theta = \int e^{+i\vec{k}\cdot\vec{x}'} \, dQ \quad .$$

The integrals on the left are taken over all of physical space while those on the right are taken over all of wave-vector space.

The magnitude of the vector $\vec{k} = (k_1, k_2, k_3)$ can be loosely considered as an inverse measure of eddy size in the sense that 2π/eddy length in the i direction $\approx k_i$. The transformation into wave-vector space thus brings together, into a small volume in this space, all eddies in the flow of a given size and shape independent of where they may occur in physical space. The largest eddies in the flow are

transformed so that they reside near the origin in wave-vector space, while the smallest eddies occupy regions far from the origin in that space. Thus, within the concept of flow of kinetic energy in wave-vector space, energy is considered to start with the largest eddies near the origin and flows outward toward the region occupied by the smallest eddies where, by the action of viscosity, it is eventually dissipated as heat. In this manner energy is removed from the gross motion of the fluid as the eddies are reduced in size and reappear more remote from the origin. This process cannot continue indefinitely without supplying energy to the flow, mainly to the larger eddies. The role of the body force per unit mass in Equation 2 is to supply this energy from the gravitational field. This concept will be used later in an attempt to reduce to manageable form, the convolution integrals which will soon appear.

The transformation is straightforward and the non-dimensional equations in wave-vector space are

$$\frac{\partial}{\partial t} dS + i \, Re \, k_j \, dZ_j = - i \, Re \int k'_j \, dZ_j (k' - k) \, dS(k') \quad . \tag{11}$$

$$\frac{\partial}{\partial t} dZ_i + k^2 \, dZ_i + \frac{1}{3} k_i k_j \, dZ_j = - i \, Re \int k'_j \, dZ_j (k' - k) \, dZ_i (k') \, . \tag{12}$$

$$\frac{\partial}{\partial t} dQ + \frac{\varkappa}{\nu} k^2 \, dQ = - i \, Re \int k'_j \, dZ_j (k' - k) \, dQ(k')$$

$$- i \, Re \left(\frac{\gamma - 1}{\gamma} \right) \frac{dZ_3}{H} \quad . \tag{13}$$

$$dP = dS + dQ \quad . \tag{14}$$

Adding Equations 11 and 13 we have a useful auxiliary equation

$$\gamma \frac{\varkappa}{\nu} k^2 \, dQ + i \, \gamma Re \, k_j \, dZ_j = \frac{Re}{H} dZ_3 \quad . \tag{15}$$

In these equations a typical length in the flow, r_o, is taken as the unit of length; a typical velocity, v_o, becomes the unit of velocity. The Reynolds number, Re, is given by $r_o v_o / \nu$. The unit of time becomes r_o^2 / ν. Therefore, all quantities in the set of equations immediately above are non-dimensional, including k which is measured in units of $1/r_o$.

SECTION 4

DECOMPOSITION OF THE VELOCITY

Moyal has shown that the component of the vector $d\vec{Z}$ in wave-vector space which is parallel to the vector $\vec{k}$ is associated with compression waves and random noise (irrotational flow). The components of $d\vec{Z}$ which are transverse to $\vec{k}$ are associated with the eddy turbulence (solenoidal flow). In order to segregate these two kinds of motion, following Moyal, we define dZ_k as the component of dZ along k and $dZ_i^{(t)}$ as the ith component of $d\vec{Z}^{(t)}$ a vector transverse to $\vec{k}$. This may be written

$$dZ_i = dZ_i^{(t)} + \frac{k_i}{\kappa} dZ_k \quad . \qquad (16)$$

Note that $k_j \, dZ_j = k \, dZ_k$ and $k_j \, dZ_j^{(t)} = 0$.

If we define the quantity $dC_i(X)$ by

$$dC_i(X) = -i \operatorname{Re} \int_{k'} k'_j \, dZ_j(k'-k) \, dX_i(k') \quad ,$$

then we have

$$dC_i(X) = dC_i^{(t)}(X) + \frac{k_i}{k} dC_k(X) \quad ,$$

where dX is any one of the spectra.

If Equation 16 is substituted into Equation 12 and both sides are multiplied by k_i/k the result is

$$\frac{\partial}{\partial t} dZ_k + \frac{4}{3} k^2 dZ_k = dC_k(Z) \quad . \qquad (17)$$

If Equation 12 is multiplied by $(\delta_{ij} - k_i k_j/k^2)$ the result is

$$\frac{\partial}{\partial t} dZ_j^{(t)} + k^2 dZ_j^{(t)} = dC_j^{(t)}(Z) \quad . \tag{18}$$

The quantity $\delta_{ij} - k_i k_j / k^2$ is called the projection operator and has the property of converting a vector in k-space into components normal to $\vec{k}$. That is, if A_j is a vector, the vector $\overline{A_i}$ is normal to $\vec{k}$ when

$$\overline{A_i} = A_j \left(\delta_{ij} - k_i k_j / k^2 \right) \quad .$$

We now have two equations which result from the original momentum equation. One of these equations governs the behavior of the spectra of the irrotational velocity (Equation 17); the other governs the behavior of the spectra of eddy turbulence (Equation 18).

SECTION 5

LINEARIZATION

The non-linear effects represented by the convolution integrals will be partially taken into account by assuming that the largest eddies in the flow (those for which $k = 0$) are the principal agents in the convective process. It is possible to decompose the convolution integrals into the following form:

$$\int k'_e \, dZ_e(k'-k) \, dZ_j(k') \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) = \int_{k' \neq k} k'_e \, dZ_e(k'-k) \, dZ_j(k') \cdot \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) + k_e \, dZ_e(0) \, dZ_i^{(t)}(k)$$

and

$$\int k'_e \, dZ_e(k'-k) \, dZ_j(k') \, k_j/k = \int_{k' \neq k} k'_e \, dZ_e(k'-k) \, dZ_j(k') \, k_j/k + k_e \, dZ_e(0) \, dZ_k(k)$$

where we have used the relations $dZ_i^{(t)}(k) = \left(\delta_{ij} - k_i k_j/k^2\right) dZ_j(k)$ and $k_i \, dZ_i = k \, dZ_k$.

The last term on the right in these equations represents the non-linear interaction of the largest eddies with the eddies of all other sizes and is the part of the integral which corresponds to the spherical shell in wave-vector space for which $k' = k$. The non-linear effects associated with the first term on the right-hand side are disregarded for the present study. Only the largest eddies are coupled to the buoyant force.

The equations for the spectra of velocity become

$$\frac{\partial}{\partial t} dZ_i^{(t)} + k^2 dZ_i^{(t)} = - i \, Re \, k_e \, dZ_e(0) \, dZ_i^{(t)}(k) \tag{19}$$

$$\frac{\partial}{\partial t} dZ_k + \frac{4}{3} k^2 dZ_k = - i \, Re \, k_e \, dZ_e(0) \, dZ_k(k) \quad . \tag{20}$$

The time dependence of the quantity $k_e \, dZ_e(0)$ is found by letting $k = 0$ in the above equations

$$\frac{\partial}{\partial t} dZ_i^{(t)}(0) = 0$$

$$\frac{\partial}{\partial t} dZ_k(0) = 0 \quad .$$

The equations may be integrated to give

$$dZ_i^{(t)}(0, t) = dZ_i^{(t)}(0, 0)$$

$$dZ_k(0, t) = dZ_k(0, 0) \quad .$$

The resulting inhomogeneous equations for the spectra of the solenoidal velocity are linear* where the relation $k_e dZ_k = k dZ_k$ has been used again

$$\frac{\partial}{\partial t} dZ_i^{(t)} + \left[k^2 + i \, Re \, k \, dZ_k(0, 0)\right] dZ_i^{(t)} = 0 \quad .$$

The equation for the spectra of the irrotational velocity is identical except that the term k^2 is replaced by $4k^2/3$.

The solutions are

$$dZ_i^{(t)}(k, t) = dZ_i^{(t)}(k, 0) \, e^{-k^2 t} \exp\left[- i \, Re \, k \, dZ_k(0, 0) \, t\right] \tag{21}$$

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*If the bouyant force term is not equated to the pressure gradient and carried along to this point the resulting linear equations have time-dependent coefficients.

$$dZ_k(k,t) = dZ_k(k,0)\, e^{-\frac{4}{3}k^2 t} \exp\left[-i\, Re\, k\, dZ_k(0,0)\, t\right] \quad . \tag{22}$$

Note that $dZ_i^{(t)}(k,0) = \left(\delta_{ij} - k_i k_j/k^2\right) dZ_j(k,0)$

$$k\, dZ_k(k,0) = k_j\, dZ_j(k,0) \quad .$$

The total velocity spectra are from Equation 16:

$$dZ_i(k,t) = dZ_i^{(t)}(k,t) + (k_i/k)\, dZ_k(k,t)$$

$$= dZ_j(k,0) \exp\left[-i\, Re\, k_j\, dZ_j(0,0)\, t\right]$$

$$\left[\left(\delta_{ij} k^2 - k_i k_j\right) \frac{e^{-k^2 t}}{k^2} + k_i k_j \frac{e^{-\frac{4}{3}k^2 t}}{k^2}\right] \quad . \tag{23}$$

The first factor involving the exponential translates the resulting inversion in physical space so that each x_j is replaced by $x_j - Re\, dZ_j(0,0)t$. The coordinate system in which the motion is most conveniently described is one which is moving with non-dimensional velocity $Re\, dZ_j(0,0)$ in each of the x_j directions. The factors $\delta_{ij}k^2 - k_i k_j$ and $k_i k_j$ become differential operators on the inverse of the functions $dZ_j(k,0)\, e^{-k^2 t}/k^2$ and $dZ_j(k,0)\, e^{-4/3\, k^2 t}/k^2$. As will be shown later, for certain initial conditions, the differential operators cause vortex and torus (or "smoke ring") formation.

If Equations 13 and 15 are combined, the energy equation becomes*

$$\frac{\partial}{\partial t} dQ + \gamma \frac{\kappa}{\nu} k^2\, dQ = -i\, Re\, k_j\, dZ_j(0,0)\, dQ - i(\gamma - 1)\, Re\, k_j\, dZ_j \tag{24}$$

*Attempts to solve the continuity equation for density yielded integrals which could not be evaluated.

with solution

$$dQ(k,t) = \exp\left(-i\,Vt - \gamma \frac{\varkappa}{\nu} k^2 t\right) \left\{ dQ(k,0) - i(\gamma - 1)\,\mathrm{Re} \int_0^t k_j\, dZ_j(k,t') \right.$$

$$\left. \exp\left[-i\,Vt' + \gamma \frac{\varkappa}{\nu} k^2 t'\right] dt' \right\} \quad ,$$

where we have defined V by

$$V = \mathrm{Re}\, k_j\, dZ_j(0,0) \quad .$$

From Equation 23 with the translating term as a factor we have

$$k_j\, dZ_j(k,t) = k_j\, dZ_j(k,0)\; e^{-\frac{4}{3} k^2 t} \exp(-i\,Vt) \quad .$$

So that

$$dQ(k,t) = \exp(-i\,Vt) \left\{ dQ(k,0)\; e^{-\gamma \frac{\varkappa}{\nu} k^2 t} - i\, \frac{(\gamma - 1)\,\mathrm{Re}\, k_j\, dZ_j(k,0)}{\left(\gamma \frac{\varkappa}{\nu} - \frac{4}{3}\right) k^2} \right.$$

$$\left. \cdot \left[\exp\left(-\frac{4}{3} k^2 t\right) - \exp\left(-\gamma \frac{\varkappa}{\nu} k^2 t\right) \right] \right\} \quad . \qquad (25)$$

To insure compatible initial conditions of velocity and temperature we use Equation 15 in the form

$$dQ(k,0) = \frac{\mathrm{Re}}{\gamma \frac{\varkappa}{\nu}} \left(\frac{1}{H} - i\,\gamma k_j\right) dZ_j(k,0)/k^2 \quad ; \qquad (26)$$

substituting into Equation 25 we have for the spectrum of the temperature function

$$dQ(k,t) = \mathrm{Re}\, dZ_j(k,0)\, \exp(-i V t) \left\{ \left[\frac{Pr}{\gamma H} - i\, Pr\, k_j + \frac{i(\gamma - 1)k_j}{\left(\gamma \frac{\chi}{\nu} - \frac{4}{3}\right)} \right] \frac{e^{-\gamma \frac{\chi}{\nu} k^2 t}}{k^2} - \frac{i(\gamma - 1)k_j}{\left(\gamma \frac{\chi}{\nu} - \frac{4}{3}\right)} \frac{e^{-\frac{4}{3} k^2 t}}{k^2} \right\} \tag{27}$$

where $Pr = \nu/\chi$, the Prandtl number.

Again, the term $\exp(-i V t)$ indicates that a moving coordinate system is the convenient frame of reference for a description of the temperature.

This will be inverted in the next section for a specific initial velocity distribution.

The density function may be found directly from the temperature function and the equation of state.

SECTION 6

QUANTITATIVE FEATURES OF THE SOLUTIONS FOR THE FIREBALL PROBLEM

SELECTION OF INITIAL CONDITIONS

Shortly after the fireball has come to pressure equilibrium it is moving upward due to ballistic and buoyant forces. The velocity is likely to be predominately in the upward direction and to be largest at the center and reducing in magnitude toward the edges of the visible fireball. We assume that the initial motion can be represented by a single component of velocity in the z(or x_3) direction. The upward initial velocity is assumed to follow a Guassian distribution so that the initial conditions are

$$v_3(\vec{x}, 0) = v_{30}\, e^{-r^2/4b} \quad , \quad v_1(\vec{x}, 0) = v_2(\vec{x}, 0) = 0 \quad , \tag{28}$$

where v_{30} is the upward velocity at the center and b is a parameter governing the rate at which the velocity reduces toward the edge of the fireball. The origin of the stationary coordinate system now requires definition and is taken at the center of the fireball at $t = 0$. The quantities r and b are non-dimensional.* When $r = 2\sqrt{b}$ the velocity has dropped by a factor of e below its value at the center of the fireball. Transformation of the initial conditions gives

$$dZ_3(k, 0) = v_{30}\left(\frac{b}{\pi}\right)^{3/2} e^{-k^2 b} \tag{29}$$

$$dZ_3(0, 0) = v_{30}\left(\frac{b}{\pi}\right)^{3/2} \quad . \tag{30}$$

THE VELOCITY FIELD

The inversion of Equation 23 with the initial conditions specified above in a frame of reference moving upward with velocity $\mathrm{Re}\,v_{30}(b/\pi)^{3/2}$ is (see Appendix C)

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*Normally we will choose $b = 1/4$ so that the velocity has reduced by a factor e when $r = 1$.

$$v_i(\vec{x},t) = -2\sqrt{\pi}\, v_{30} b^{3/2} \left[\left(\delta_{i3} \nabla^2 - \frac{\partial^2}{\partial x_i \partial x_3} \right) \frac{1}{r} \operatorname{erf} \frac{r}{2\sqrt{t+b}} + \frac{\partial^2}{\partial x_i \partial x_3} \frac{1}{r} \operatorname{erf} \frac{r}{2\sqrt{\frac{4}{3}t+b}} \right] . \tag{31}$$

The moving frame of reference is introduced as a convenience resulting from the factor $\exp[-i\, Rek_j\, dZ_j(0, 0)]$ in the original transform of the velocity.

Performing the differentiation the velocity in the $x_i (i = 1, 2)$ direction is

$$v_i(\vec{x},t) = 2\sqrt{\pi}\, v_{30} \frac{x_i x_3}{r^5} b^{3/2} \left[T\left(\frac{r}{\tau_1}\right) - T\left(\frac{r}{\tau_2}\right) \right] , \tag{32}$$

where

$$T(u) = u^2 \operatorname{erf}'' u - 3u \operatorname{erf}' u + 3 \operatorname{erf} u$$

$$\tau_1 = 2\sqrt{t+b}$$

$$\tau_2 = 2\sqrt{4t/3+b}$$

$$\frac{\partial}{\partial u} \operatorname{erf} u = \operatorname{erf}' u .$$

The radial velocity in any horizontal plane is given by

$$v_\rho = \left(v_1^2 + v_2^2 \right)^{1/2}$$

$$= 2\sqrt{\pi}\, v_{30} \frac{x_3 \rho}{r^5} b^{3/2} \left[T\left(\frac{r}{\tau_1}\right) - T\left(\frac{r}{\tau_2}\right) \right] , \tag{33}$$

where $\rho = \left(x_1^2 + x_2^2 \right)^{1/2}$.

The velocity in the x_3 direction is

$$v_3(\vec{x},t) = -2\sqrt{\pi}\, v_{30}\, b^{3/2} \left\{ \frac{\rho^2}{r^5} T\left(\frac{r}{\tau_1}\right) + \frac{x_3^2}{r^5} T\left(\frac{r}{\tau_2}\right) + \frac{1}{r^3}\left[2S\left(\frac{r}{\tau_1}\right) + S\left(\frac{r}{\tau_2}\right)\right] \right\} \qquad (34)$$

where $S(u) = u \, \mathrm{erf}' u - \mathrm{erf}\, u$.

The functions $T(u)$ and $S(u)$ are plotted versus u in Figure 1.

Note that as $t \to 0$, $\tau_1 \to \tau_2 = \tau$ and we have

$$v_1(\vec{x},0) = v_2(\vec{x},0) = 0 \qquad , \qquad t \to 0$$

$$v_3(\vec{x},0) = -2\sqrt{\pi}\, v_{30}\, b^{3/2} \frac{1}{\tau^2 r} \mathrm{erf}''\left(\frac{r}{\tau}\right) \qquad , \qquad t \to 0 \ .$$

The derivatives of the error function are

$$\mathrm{erf}'\, u = \frac{2}{\sqrt{\pi}} e^{-u^2}$$

$$\mathrm{erf}''\, u = -\frac{4u}{\sqrt{\pi}} e^{-u^2}$$

Yielding

$$v_3(\vec{x},0) = v_{30}\, e^{-r^2/4b}$$

The initial conditions have been recovered at $t = 0$.

The horizontal radial velocity v_ρ given by Equation 33 is seen to change signs as we move from above to below the origin of the moving coordinate system. It is zero on the vertical axis and on the horizontal plane through the moving origin. It rapidly approaches zero as $r \to \infty$.

The general character of the horizontal motion is as shown in Figure 2, where the velocity is shown outward from the vertical axis above the horizontal plane containing the moving origin and toward the axis below this plane.

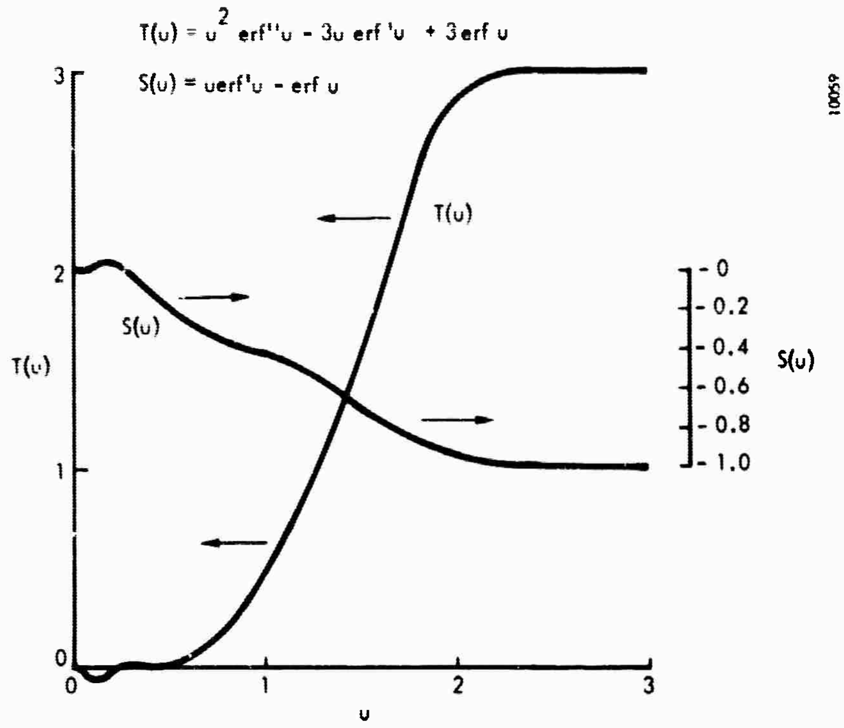

Figure 1. The function T(u) and S(u) versus u .

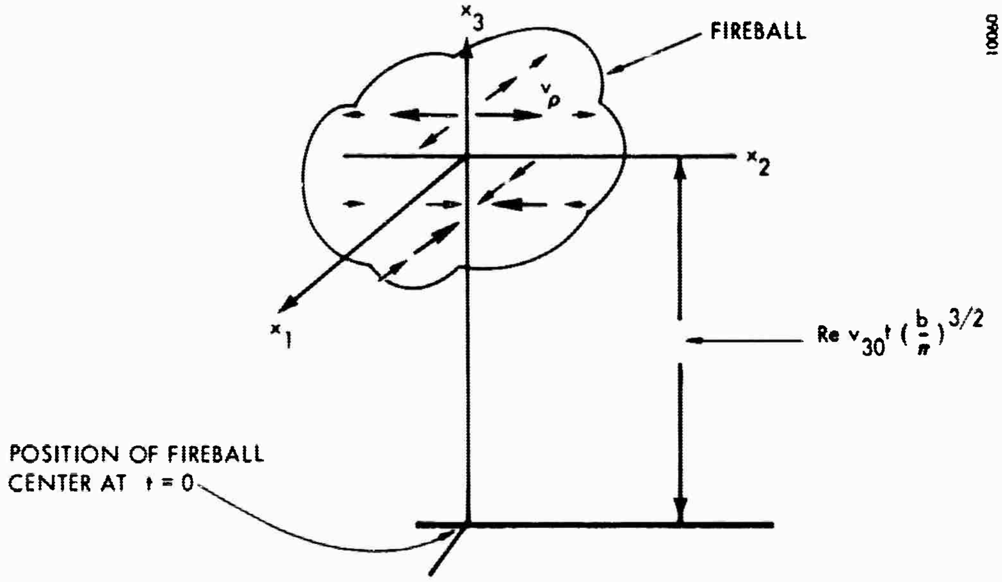

Figure 2. Configuration of the horizontal velocity in the moving frame of reference.

The vertical velocity has a more complex dependence on time and the space coordinates. In the moving frame of reference it is symmetrical about the plane $x_3 = 0$, positive near the vertical axis, and negative far from the axis for small values of time. As time increases it reduces in value near the vertical axis and increases at values of ρ which are increasingly large. Its behavior is as shown in Figure 3.

When combined with the horizontal radial velocity v_ρ a configuration in the moving frame shown in Figure 4 results. This displays the features often observed in nuclear weapons tests (and chemical explosions) where a toroid or vortex ring is formed.

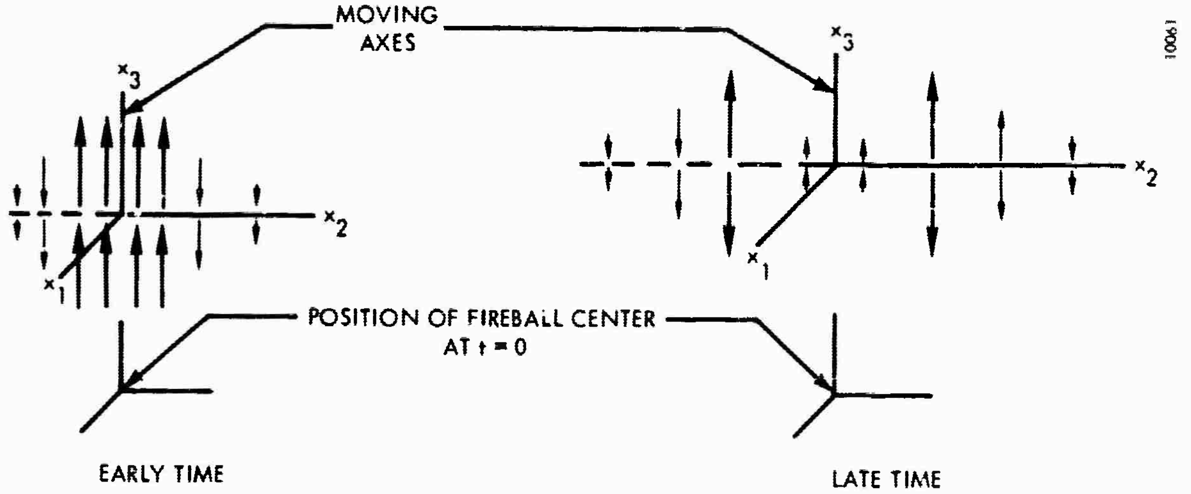

Figure 3. Configuration of the upward velocity in the moving frame of reference in the plane $x_1 = 0$.

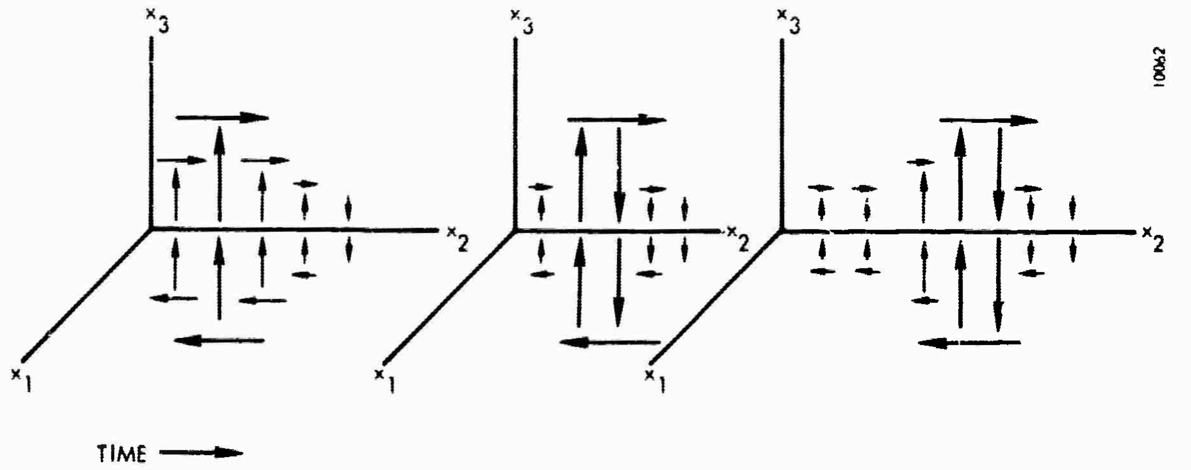

Figure 4. Velocity configuration at three different times showing vortex ring formation in a frame of reference moving upward with uniform velocity.

TEMPERATURE DISTRIBUTION

If we continue to employ an initial upward velocity which has a Gaussian distribution in space about the center of the early fireball, the temperature spectra is given by

$$dQ(k,t) = \mathrm{Re}\, v_{30} (b/\pi)^{3/2} \exp[-iVt] \left\{ \left(\frac{Pr}{\gamma H} - i\, Pr\, k_3 + \frac{i(\gamma-1)k_3}{(\gamma/Pr - 4/3)} \right) \frac{e^{-\left(\frac{\gamma}{Pr} t+b\right)k^2}}{k^2} - \left(\frac{i(\gamma-1)k_3}{(\gamma/Pr)-4/3} \right) \frac{e^{-\left(\frac{4}{3} t+b\right)k^2}}{k^2} \right\} , \quad (36)$$

where $Pr = \nu/\chi$, the Prandtl number ($\cong 0.7$ for air). The factor $\exp[-iVt]$ translates the resulting inversion so that x_3 is replaced by $x_3 - \mathrm{Re}\, dZ_3(0, 0)t$ and again the coordinate system offering the greatest utility is one moving upward with uniform non-dimensional velocity $\mathrm{Re}\, dZ_3(0, 0) = \mathrm{Re}\, v_{30}(b/\pi)^{3/2}$. If the spectra of temperature is inverted and expressed in this system the result is (Appendix C)

$$\theta = 8\sqrt{\pi}\, v_{30}\, b^{3/2}\, \mathrm{Re} \left[\frac{Pr}{\gamma H} \frac{1}{r} \mathrm{erf} \frac{r}{2\sqrt{\frac{\gamma}{Pr} t+b}} - \frac{1-4\,Pr/3}{\gamma/Pr-4/3} \frac{\partial}{\partial x_3} \left(\frac{1}{r} \mathrm{erf} \frac{r}{2\sqrt{\frac{\gamma}{Pr} t+b}} \right) - \frac{(\gamma-1)}{\gamma/Pr-4/3} \frac{\partial}{\partial x_3} \left(\frac{1}{r} \mathrm{erf} \frac{r}{2\sqrt{4t/3+b}} \right) \right] . \quad (37)$$

Performing the differentiation yields

$$\theta = 8\sqrt{\pi}\, v_{30}\, b^{3/2}\, \mathrm{Re} \left\{ \frac{Pr}{\gamma H r} \mathrm{erf} \frac{r}{\tau_A} - \frac{x_3}{\left(\frac{\gamma}{Pr} - \frac{4}{3}\right) r^3} \left[(1-4\,Pr/3)\, S\left(\frac{r}{\tau_A}\right) + (\gamma-1)\, S\left(\frac{r}{\tau_B}\right) \right] \right\} , \quad (38)$$

where $\tau_A = 2\sqrt{\frac{\gamma}{Pr} t+b}$, $\tau_B = 2\sqrt{4t/3 + b}$ and, as before, $S(u) = u\, \mathrm{erf}'\, u - \mathrm{erf}\, u$.

If we let $b = 1/4$, the length scale is established such that the initial velocity drops by a factor e at a distance r_o from the center of the fireball.

To establish the velocity scale let $t = 0$, $r \to 0$ and note that erf $x \to 2x/\sqrt{\pi}$, x small. This yields $\theta = 2\,\frac{v_{30} \text{Re Pr}}{\gamma H \tau_A}$ from Equation 38 at $r = 0$* for $b = 1/4$. Define θ_{oc} as the value of θ at $t = 0$, $r = 0$ (at the center of the fireball) so that

$$\theta_{oc} = \frac{2\, v_{30}\ \text{Re Pr}}{\gamma H} \quad .$$

Substituting into Equation 38 to remove the Reynolds number, velocity product

$$\theta = \frac{\sqrt{\pi}}{2}\, \theta_{oc} \left\{ \frac{1}{r}\, \text{erf}\, \frac{r}{\tau_A} - \frac{x_3 \gamma H}{(\gamma - 4\text{Pr}/3) r^3} \left[(1 - 4\text{Pr}/3)\, S\left(\frac{r}{\tau_A}\right) + (\gamma - 1)\, S\left(\frac{r}{\tau_B}\right) \right] \right\} \quad .$$

Along the horizontal midplane $x_3 = 0$ the temperature function is

$$\theta = \frac{\sqrt{\pi}}{2\rho}\ \theta_{oc}\ \text{erf}\left(\frac{\rho}{2\sqrt{2t + 1/4}} \right) \quad ,$$

where we have taken $\gamma = 1.4$, $\text{Pr} = 0.7$ and $\rho = (x_1^2 + x_2^2)^{1/2}$.

This function is relatively constant for small values of $\rho/2\sqrt{2t + 1/4}$ as ρ increases indicating the presence of a thin "sheet" of high temperature air in the horizontal midplane of the moving coordinate system. For small values of $\rho/2\sqrt{2t + 1/4}$ the temperature for $x_3 = 0$ is

$$\theta \cong \theta_{oc} / 2\sqrt{2t + 1/4} \quad ,$$

giving (using $e^{\theta} = T/T_o$)

$$\frac{T}{T_o} = \left(\frac{T_{oc}}{T_o} \right)^{\frac{1}{\sqrt{8t + 1}}} \quad .$$

Note that T_o is the temperature of the atmosphere far from the fireball.

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*We approach the moving origin along a horizontal path.

SECTION 7
DISCUSSION

The velocity and temperature distributions obtained in the previous sections result from an extension of the theory of turbulence for the final period of decay. The extension is brought into the theory by including that part of the non-linear terms which corresponds to coupling of the largest eddies with eddies of all other sizes. The resulting linearization, which is carried out in wave-vector space, has the effect of translating the space coordinates in the prescription of the velocity and temperature in a wave-like manner depending upon the velocity of the largest eddies in the initial motion. The resulting translational velocity is to be identified with the rise velocity of the fireball if the initial motion has only an upward component. A convenient coordinate system for describing the motion and temperature in the case of more general initial conditions is one which is moving in the direction of the velocity of the largest eddies in the initial motion.

From a physical point of view the energy source most effective in the motion of the late-time fireball is the gravitational field. Its origin is in the downward motion of those parts of the atmosphere outside the fireball. This may be shown analytically by retaining the constant buoyant force term through the transformation to wave-vector space. The term in the transformed equations which represents the buoyant force produces a downward velocity which is effective on only the largest eddies in the flow since it contains the Dirac delta function as a factor. Attempts to continue the analysis in this direction, which is tantamount to relaxing the assumption of pressure equilibrium, were unsuccessful due to the difficulty associated with the interpretation of integrals which arose containing the delta function. It was felt that formal application of the rules for handling the delta function could lead to ambiguous results from a physical point of view. For this reason, the assumption of pressure equilibrium, which states that the pressure at corresponding altitudes within the flow is equal to that in an undisturbed exponential atmosphere, was used before the transformation was applied to the momentum equation. This, in effect, removed the energy source allowing the motion to proceed in accordance with the

theory of the final period of decay of turbulent flow. The resulting effects are:

1. To remove from the solutions for the irrotational part of the velocity those terms which correspond to propagating sound waves such as those obtained in Appendix A

2. To remove a term in the energy equation which depends on the time derivative of pressure

3. To simplify those non-linear terms in the energy equation which involve space derivatives of the pressure.

Item 2 has at least removed the effect of radiating sound waves on the temperature distribution. Item 3 has an effect which is difficult to determine since, in essence, it is a linearization process brought about by the fact that in an exponential atmosphere the only space derivative of the logarithm of the pressure π_p which is non-zero is the derivative with respect to the vertical direction and it is a constant.

The overall effect on the motion of the assumptions considered above is to smooth out irregularities leading to a deterministic but certainly approximate description of the velocity and temperature. The smoothing which occurs prevents this method from providing accurate information on the problem of radar clutter due to a fireball. The method employed does provide at least qualitative information on the velocity and temperature distribution. An extension of the analysis to include the influence of irregular pressure waves may be required. A possible starting point is to employ the solutions obtained in Appendix A.

The form of the solutions obtained in this report have suggested an approach to the problem of fireball mixing involving the use of Hermite functions. Employing a Gaussian distribution of upward velocity as the initial condition the solutions found are expressed in terms of the error function and its derivatives. These derivatives as well as the initial conditions, may be written as Hermite functions which have several properties which may be useful in obtaining solutions of the fundamental equations.[15] For example, the Fourier transform of a Hermite function is also a Hermite function, convolution integrals involving Hermite functions may be performed analytically, a series of Hermite functions is capable of expressing a wide range of generalized functions.[16]

Another alternative approach and one which is usually employed in the general theory is to determine relationships among the various statistical features of the dynamics, such as correlation coefficients, but to include

better approximations to the governing equations. Some of these alternatives are considered in Appendix B. Since no direct data exist on correlation coefficients within the fireball, the spectra of velocity and temperature were felt to be more useful for this application.

Although the velocity field found for the initial conditions used here appears to represent a developing vortex ring, the temperature in the horizontal midplane of the moving coordinate system has a form which represents a vertically thin layer or membrane of high temperature air within the ring of the vortex which is dropping in temperature due to ordinary conduction. This occurs because the velocity in the center of the ring is small in the moving frame of reference and the thin layer rises while diffusion of energy occurs by heat conduction. At other positions the terms involving the velocity which appear as "source" terms in the energy equation act to increase, by convective transport, the temperature outside the fireball and reduce it on the inside. In this way the fireball "grows" by outward movement of heated air which mixes with the atmosphere. Only part of this convective effect is included in the solutions for the temperature found in this report. The non-linear term $v_i \, \partial\theta/\partial x_i$ which represents this convective effect is only partially included. The other source term represents the effect of expansion and compression on the temperature and probably is less important.

If the solution for the velocity is simply substituted into the energy equation or the continuity equation (see Footnote page 13) integrals are found which represent singular solutions. This is probably related physically to the impossibility of maintaining exact pressure equilibrium in the flow. The full understanding of this and many other features of the problem must be reserved for later investigation.

SECTION 8

CONCLUSIONS AND RECOMMENDATIONS

The conclusions obtained as the result of this study are:

1. The most directly useful results for the fireball mixing problem will be obtained from the velocity, temperature, pressure, and density rather than from solutions describing correlation coefficients involving these variables.

2. An extension of the theory of turbulence in the final period of decay may be obtained by including that part of the non-linear inertia terms which provides a translational effect. The largest eddies in the initial motion convect, without distortion, the eddies of smaller size.

3. The velocity distribution of a heated mass of air rising in the atmosphere obtained from the equations of change of a viscous, heat conducting, compressible gas show features which correspond in several respects to the actual characteristics of the late-time fireball of a nuclear detonation.

Recommendations for further work are:

1. A more detailed comparison of the results obtained in this report with experimental data should be carried out. Different kinds of initial conditions should be employed.

2. Use of machine computation to obtain solutions of the equations of change in spectral form should be carried out. This approach would begin with an analysis of the one-dimensional case.

3. A study of the general theory of turbulent flow as related to the problem of the late-time fireball should be undertaken. This study should include:

   a. Attempts to employ the solutions of Appendix B as a starting point to obtain information useful for radar clutter problems.

b. A study of methods of generalizing the linearization process employed in this report.

c. An investigation of the use of higher transcendental functions, such as Hermite functions using the theory of Pansions.[15]

APPENDIX A

ASYMPTOTIC SOLUTIONS OF THE LINEAR EQUATIONS

In non-dimensional form, after Fourier transformation with respect to the independent space variables, the equations of Moyal are of the form

$$\frac{d}{dt}\left|dZ_i^{(t)}\right| + k^2\, dZ_i^{(t)} = \mathrm{Re}\, dC_i^{(t)}(Z)$$

$$\frac{d}{dt}\left|dZ_k\right| + \frac{4}{3}k^2\, dZ_k + \mathrm{Re}\frac{i\,a^2\,k}{\gamma}(dS + dQ) = \mathrm{Re}\, dC_k(Z)$$

$$\frac{d(dS)}{dt} + i\,k\,\mathrm{Re}\, dZ_k = \mathrm{Re}\, dC(S)$$

$$\frac{d(dQ)}{dt} + i\,k\,(\gamma - 1)\,\mathrm{Re}\, dZ_k + \gamma\,\frac{\varkappa}{\nu}\,\nabla^2 Q = \mathrm{Re}\, dC(Q) \quad .$$

As such, as is usually the case, successive approximation procedures are suggested for the two limiting cases of vanishing Reynolds number and for large Reynolds number. Moyal himself wrote down a solution for the homogeneous case neglecting the non-linear convolution integrals which he claimed was accurate to terms of order 1/ .. In dimensionless form his solutions for a zero mean velocity assume the form

$$dZ_i^{(t)} = dZ_i^{(t)}\Big|_o\, e^{-k^2 t}$$

$$dZ_k = \frac{1}{2}\, e^{-\frac{\alpha}{\nu}k^2 t}\left\{\left[dZ_k\Big|_o + \frac{\alpha}{\gamma}(dS_o + dQ_o)\right] e^{-iaktRe} + \left[dZ_k\Big|_o - \frac{a}{\gamma}(dS_o + dQ_o)\right] e^{+iaktRe}\right\}$$

$$dS = \frac{1}{2a}\, e^{-\frac{\alpha}{\nu}k^2 t}\left\{ dZ_k\Big|_o - \frac{a}{\gamma}(dS_o + dQ_o)\, e^{-iaktRe}\right.$$

$$- \left[dZ_k \Big|_o - \frac{a}{\gamma} (dS_o + dQ_o) \right] e^{iaktRe} \Big\}$$

$$+ \frac{1}{\gamma} e^{-\frac{\varkappa}{\nu} k^2 t} \left[(\gamma - 1)\, dS_o + dQ_o \right]$$

$$dQ = \frac{\gamma - 1}{2a} e^{-\frac{\alpha}{\nu} k^2 t} \Big\{ \left[dZ_k \Big|_o + \frac{a}{\gamma} (dS_o + dQ_o) \right] e^{-iaktRe}$$

$$- \left[dZ_k \Big|_o - \frac{a}{\gamma} (dS_o + dQ_o) \right] e^{iaktRe} \Big\}$$

$$- \frac{1}{\gamma} e^{-\frac{\varkappa}{\nu} k^2 t} \left[(\gamma - 1)\, dS_o + dQ_o \right]$$

where

$$\alpha = \frac{1}{2} \left[\frac{4}{3} \nu + (\gamma - 1)\, \varkappa \right]$$

$$\frac{t_{actual}}{r_o^2/\nu} = t_{non\text{-}dim}$$

$$k_{actual} = k_{non\text{-}dim}/r_o$$

$$a_{actual} = v_o\, a_{non\text{-}dim}$$

$$dZ_{actual} = v_o\, dZ_{non\text{-}dim}$$

$$Re = \frac{v_o r_o}{\nu} .$$

We have noticed, however, that this approximation has the following remarkable property; namely, it is an approximation which gives the correct sum and triple product of the three roots of the secular equation for the last three coupled equations involving the longitudinal components, but it does not correctly give the sum of the pair-wise products of the roots. As such, however, it would form a good basis for computing a more accurate approximation to each root by, for example, Newton's method. Since we were, however, interested in trying to understand at least some of the simpler aspects of the nonlinear phenomena, we did not pursue this matter further.

The non-dimensional form of these equations does, however, also suggest a successive approximation procedure which in principle would yield information for large Reynolds numbers by standard techniques. It is to be noticed that the first equation does not involve the Reynolds number on the left-hand side while the coupled set are in a form to which methods have been developed for finding a solution to the homogeneous equations as they are, for example, described in Chapter 6 of The Theory of Ordinary Differential Equations, by Coddington and Levison, McGraw-Hill, 1955. Once such a fundamental set of solutions were known along with the trivial solutions for the homogeneous uncoupled transverse equations, the convolution integrals on the right-hand side of Moyal's equations could be evaluated. The right-hand side of the equations could then be thought of as known and a first approximation to the non-linear effects found as follows: The uncoupled equations for the transverse components could be solved directly by the method of variation of parameters and added to the trivial general solution to the homogeneous part of these equations. If the right-hand side of the coupled equations describing the component along k is denoted by the vector $\underline{b(t,k)}$ and if the fundamental matrix for a large value of the Reynolds number, found by solving the homogeneous coupled equations by a procedure we will outline, were denoted by $\Phi(t,k)$ then the vector function $\underline{\varphi}(t,k)$ defined by the relation [Equation 3.1 page 74 of Coddington and Levinson]

$$\underline{\varphi}(t,k) = \Phi(t,k) \int_{t_0}^{t} \Phi^{-1}(s,k)\, \underline{b(s,k)}\, ds$$

would be a solution of the non-homogeneous equations vanishing for t_0. To this could be added the general solution of the homogeneous equations already found in order to have a first-order approximation to the non-linear effects.

Unfortunately, time did not permit us to explore this mode of attack in detail although we did make a beginning in this direction. Since there is some possibility that work in this area will be funded on a long-range basis in the future, it appears to us to be worthwhile to record the progress obtained to date by this approach. If for simplicity we let $\beta = \varkappa/\nu = \frac{1}{Pr}$, where Pr is the Prandtl number, then the coupled homogeneous equations of Moyal can be written in the matrix form

$$\begin{pmatrix} \frac{d}{dt}(dZ_k) \\ \frac{d}{dt}(dS) \\ \frac{d}{dt}(dQ) \end{pmatrix} = \mathrm{Re}\left[\begin{pmatrix} 0 & -\frac{i a^2 k}{\gamma} & -\frac{i a^2 k}{\gamma} \\ -ik & 0 & 0 \\ -i(\gamma-1)k & 0 & 0 \end{pmatrix} + \frac{1}{\mathrm{Re}}\begin{pmatrix} -\frac{4}{3}k^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\gamma\beta k^2 \end{pmatrix}\right]\begin{pmatrix} dZ_k \\ dS \\ dQ \end{pmatrix}$$

That is in the general matrix form

$$\frac{d\underline{x}}{dt} = \mathrm{Re}\left(A_o + \frac{1}{\mathrm{Re}} A_1\right)\underline{x} \quad .$$

If B_o is a non-singular matrix such that $B_o^{-1} A_o B_o = D_o$ where D_o is a diagonal matrix with distinct roots that can be found, as we will show to be the case for Moyal's equations, then the transformation $\underline{x} = B_o \underline{y}$ leads to a system of the form

$$\underline{y}' = \mathrm{Re}\left[D_o + \frac{1}{\mathrm{Re}} A_1^{(\prime)}\right]\underline{y} \quad . \tag{A-1}$$

For Moyal's equations the situation is particularly simple for it is possible to find a matrix B_o with the above properties which is, in fact, independent of the time. In this situation $A_1^{(\prime)} = B_o^{-1} A_1 B_o$. For simplicity we will again denote this by A_1.

We will now briefly describe the method used to obtain a formal solution to Equation A-1. This description is, in reality, a specialization of that given in Chapter 6 of Coddington and Levinson, but it seems worthwhile to include it, for the case at hand is considerably simpler than the general case treated by them. We seek a solution for $\underline{y}$ in the form

$$\underline{y} = P \exp Q \quad , \tag{A-2}$$

where $P(t)$ and $Q(t)$ are matrices, $Q(t)$ being assumed diagonal. For $P(t)$ we assume an expansion of the form

$$P(t) = P_0(t) + \frac{1}{\mathrm{Re}} P_1(t) + \frac{1}{\mathrm{Re}^2} P_2(t) + \ldots .$$

while for Equation A-1 it suffices to take $Q = \mathrm{Re}\, Q_0(t) + Q_1(t)$ where, as we shall show, $Q_o'(t) = D_o$, a constant matrix and where Q_1 is also a diagonal matrix by assumption.

Performing the indicate differentiation and substituting into Equation A-1 we obtain after cancellation of the exponential factor

$$\left[P_o' + \frac{1}{Re} P_1' + \frac{1}{Re^2} P_2'\right] + \left[P_o + \frac{1}{Re} P_1 + \frac{1}{Re^2} P_2 + \ldots\right]$$

$$\left[Re\, Q_o' + Q_1'\right] = Re\left[\left(D_o + \frac{1}{Re} A_1\right)\left[P_o + \frac{1}{Re} P_1 + \frac{1}{Re^2} P_2 + \ldots\right]\right] \quad .$$

Rearranging according to decreasing powers of the Reynolds number, the above becomes

$$Re\; P_oQ_o' - D_o P_o + (Re)^0\left[P_o' + P_1 Q_o' + P_oQ_1' - D_oP_1 - A_1P_o\right]$$

$$+ \frac{1}{Re}\left[P_1' + P_2 Q_o' + P_1 Q_1' - D_oP_2 - A_1 P_1\right]$$

$$+ \frac{1}{Re^2}\left[P_2' + P_3 Q_o' + P_2 Q_2' - D_oP_3 - A_1 P_2\right] + \ldots = 0 \quad .$$

By equating the various powers of the Reynolds number to zero we obtain the following infinite set of matrix equations:

$$P_oQ_o' - D_oP_o = 0$$

$$P_o' + P_1Q_o' + P_oQ_1' - D_oP_1 - A_1 P_o = 0$$

$$P_1' + P_2Q_o' + P_1Q_1' - D_oP_2 - A_1 P_1 = 0$$

$$P_2' + P_3Q_o' + P_2Q_1' - D_oP_3 - A_1 P_2 = 0$$

. . . . .

The first of these is easily satisfied by taking $P_o = I$ = the identity matrix which, of course, implies that $P_o' = 0$ and setting $Q_o' = D_o$. To satisfy the second equation it is first rewritten in the form

$$P_1D_o - D_oP_1 = A_1 - Q_1'$$

in which use has been made of the information already obtained. Now it is clear that there are no diagonal terms on the left-hand side of the above equation. Thus, the diagonal terms on the right-hand side must cancel so that if we let

$$A_1 = (a_{ij})$$

we see that since Q_1 is diagonal, by assumption $Q_1' = (a_{ij}\,\delta_{ij})$ where δ_{ij} is the Kronecker delta equal to one for equal indices and equal to zero for different indices. For $i \neq j$ we then find that if $P_1 = (p_{ij})$, $i \neq j$ then

$$p_{ij} = a_{ij}/(\lambda_i - \lambda_j) \quad ,$$

where λ_i, λ_j are distinct roots of $D_o = (\lambda_i\,\delta_{ij})$. In order to determine the diagonal terms of P_1, P_1 is replaced by $P_1^* = P_1 + D_1$ where D_1 is as yet an unknown diagonal matrix and where P_1 is assumed to have a zero diagonal. P_1^* is clearly still a solution of the second equation along with P_1 for such an arbitrary D_1. To determine D_1 we use the next equation rewritten in the form

$$P_1' + D_1' + P_1 Q_1' - A_1 P_1 = A_1 D_1 - D_1 Q_1' + P_2 D_o - D_o P_2 \quad .$$

In this form there are no diagonal terms on the right-hand side and since P_1' had a zero diagonal, it follows that the diagonal matrix D_1' must be given by the diagonal terms of $A_1 P_1 - P_1 Q_1'$. This same process can be continued to higher order but we shall not carry it out. We will continue our comments on the general theory with the observation that for the equations of Moyal D_o, A_1 hence Q_o', P_1, Q_1' and D_1' all turn out to be matrices independent of the time. This means that the above procedure will produce in this particular case a particular simple type solution of the form

$$\underline{y} = \left[I + \frac{1}{Re}(P_1 + D_1 t)\right] \exp\left[Re(\lambda_i\,\delta_{ij}) + (a_{ij}\,\delta_{ij})\right] t \quad ,$$

or

$$\underline{x} = B_o \exp\left[Re\,(\lambda_i\,\delta_{ij}) + (a_{ij}\,\delta_{ij}) t\right]$$

$$+ \frac{1}{Re}(B_o P_1 + B_o D_1\, t) \exp\left[Re\,(\lambda_i\,\delta_{ij}) + (a_{ij}\,\delta_{ij})\right] t \quad .$$

To apply this theorm to the equation of Moyal involving the component along $\underline{k}$, it is first necessary to determine the eigenvalues of A_o from the equation

$$\det(\lambda I - A_o) = 0$$

or

$$\begin{vmatrix} \lambda & \dfrac{i\,a^2k}{\gamma} & \dfrac{i\,a^2k}{\gamma} \\ ik & \lambda & 0 \\ ik(\gamma-1) & 0 & \lambda \end{vmatrix} = 0 \;.$$

A simple computation yields the three roots

$$\lambda_o = 0 \quad , \qquad \lambda_1 = -i\,a\,k \quad , \qquad \lambda_2 = i\,a\,k \quad .$$

We note the two relations $\lambda_1^2 = \lambda_2^2 = -a^2k^2$, $i\lambda_1 - i\lambda_2 = 2\,a$.

In order to determine the matrix B_o with the property that

$$B_o^{-1} A_o B_o = D_o$$

we use the remark on page 176 of Coddington and Levinson and take for the k-th column of B_o a multiple of the cofactors of a row of $A_o - \lambda_k I$. Specifically, we chose for B_o the matrix

$$B_o = \begin{pmatrix} 0 & i\lambda_1 & i\lambda_2 \\ -1 & k & k \\ 1 & k(\gamma-1) & k(\gamma-1) \end{pmatrix} = \begin{pmatrix} 0 & a\,k & -a\,k \\ -1 & k & k \\ 1 & k(\gamma-1) & k(\gamma-1) \end{pmatrix} \;.$$

We next verify the relation $A_o B_o = B_o D_o$ by direct computation, making use of the simple properties possessed by the characteristic roots noted previously. Thus

$$A_o B_o = \begin{pmatrix} 0 & -\dfrac{i\,a^2k}{\gamma} & -\dfrac{i\,a^2k}{\gamma} \\ -ik & 0 & 0 \\ -ik(\gamma-1) & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & i\lambda_1 & i\lambda_2 \\ -1 & k & k \\ 1 & k(\gamma-1) & k(\gamma-1) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\dfrac{i\,a^2k^2}{\gamma} - i\,\dfrac{a^2k^2}{\gamma}(\gamma-1) & -\dfrac{i\,a^2k^2}{\gamma} - i\,\dfrac{a^2k^2}{\gamma}(\gamma-1) \\ 0 & k\,\lambda_1 & k\,\lambda_2 \\ 0 & \lambda_1\,k(\gamma-1) & \lambda_2\,k(\gamma-1) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -i\,a^2k^2 & -\,i\,a^2k^2 \\ 0 & \lambda_1 k & \lambda_2 k \\ 0 & \lambda_1\,k(\gamma-1) & \lambda_2\,k(\gamma-1) \end{pmatrix}$$

$$
= \begin{pmatrix} 0 & i\lambda_1^2 & i\lambda_1^2 \\ 0 & \lambda_1 k & \lambda_2 k \\ 0 & k(\gamma-1)\lambda_1 & k(\gamma-1)\lambda_2 \end{pmatrix}
$$

Similarly

$$
B_oD_o = \begin{pmatrix} 0 & i\lambda_1 & i\lambda_2 \\ -1 & k & k \\ 1 & k(\gamma-1) & k(\gamma-1) \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}
$$

$$
= \begin{pmatrix} 0 & i\lambda_1^2 & i\lambda_2^2 \\ 0 & k\lambda_1 & k\lambda_2 \\ 0 & k\lambda_1(\gamma-1) & k\lambda_2(\gamma-1) \end{pmatrix} .
$$

We also note that

$$
B_oB_o^{-1} = \begin{pmatrix} 0 & i\lambda_1 & i\lambda_2 \\ -1 & k & k \\ 1 & k(\gamma-1) & k(\gamma-1) \end{pmatrix} \left(\frac{\begin{matrix} 0 & -2a\,k^2(\gamma-1) & 2a\,k^2 \\ k\gamma & -i\,\lambda_2 & -i\,\lambda_2 \\ -k\gamma & i\,\lambda_1 & i\,\lambda_1 \end{matrix}}{2a\,k^2\gamma} \right)
$$

$$
= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = B_o^{-1}B_o .
$$

These relations make explicit the proper form for B_o and B_o^{-1} for both right and left matrix multiplication. We next complete $A_1^{(1)} = B_o^{-1}\,A_1B_o$ in two successive steps.

$$A_1B_o = \begin{pmatrix} -\frac{4}{3}k^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\gamma\beta k^2 \end{pmatrix}\begin{pmatrix} 0 & ak & -ak \\ -1 & k & k \\ 1 & k(\gamma-1) & k(\gamma-1) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\frac{4}{3}ak^3 & \frac{4}{3}ak^3 \\ 0 & 0 & 0 \\ -\gamma\beta k^2 & -\gamma\beta k^3(\gamma-1) & -\gamma\beta k^3(\gamma-1) \end{pmatrix} .$$

We next note that B_o^{-1} can be written as:

$$B_o^{-1} = \begin{pmatrix} 0 & -\left(\frac{\gamma-1}{\gamma}\right) & \frac{1}{\gamma} \\ \frac{1}{2ak} & \frac{1}{+2k\gamma} & \frac{1}{2k\gamma} \\ -\frac{1}{2ak} & \frac{1}{2k\gamma} & \frac{1}{2k\gamma} \end{pmatrix} .$$

Then we find $A_1^{(1)} = B_o^{-1}A_oB_o$

$$= \begin{pmatrix} 0 & -\frac{(\gamma-1)}{\gamma} & \frac{1}{\gamma} \\ \frac{1}{2ak} & \frac{1}{2k\gamma} & \frac{1}{2k\gamma} \\ -\frac{1}{2ak} & \frac{1}{2k\gamma} & \frac{1}{2k\gamma} \end{pmatrix}\begin{pmatrix} 0 & -\frac{4}{3}ak^3 & \frac{4}{3}ak^3 \\ 0 & 0 & 0 \\ -\gamma\beta k^2 & -\gamma\beta k^3(\gamma-1) & -\gamma\beta k^3(\gamma-1) \end{pmatrix}$$

$$= \begin{pmatrix} -\beta k^2 & -\beta k^3(\gamma-1) & -\beta k^3(\gamma-1) \\ -\frac{\beta k}{2} & -\frac{2}{3}k^2 - \frac{\beta k^2(\gamma-1)}{2} & \frac{2}{3}k^2 - \frac{\beta k^2(\gamma-1)}{2} \\ -\frac{\beta k}{2} & \frac{2}{3}k^2 - \frac{\beta k^2(\gamma-1)}{2} & -\frac{2}{3}k^2 - \frac{\beta k^2(\gamma-1)}{2} \end{pmatrix} .$$

Therefore

$$Q_1 = \begin{pmatrix} -\beta k^2 t & 0 & 0 \\ 0 & \left[-\frac{2}{3}k^2 - \frac{\beta k^2}{2}(\gamma-1)\right]t & 0 \\ 0 & 0 & \left[-\frac{2}{3}k^2 - \frac{\beta k^2}{2}(\gamma-1)\right]t \end{pmatrix}$$

$$Q_o = \begin{pmatrix} 0 & 0 & 0 \\ 0 & iakt & 0 \\ 0 & 0 & iakt \end{pmatrix}$$

and

$$e^Q = e^{ReQ_o + Q_1}$$

$$= \exp\left[Re \begin{pmatrix} 0 & 0 & 0 \\ 0 & -iakt & 0 \\ 0 & 0 & iakt \end{pmatrix} + \begin{pmatrix} \beta k^2 t & 0 & 0 \\ 0 & \left[-\frac{2}{3}k^2 - \frac{\beta k^2}{2}(\gamma-1)\right]t & 0 \\ 0 & 0 & \left[-\frac{2}{3}k^2 - \frac{\beta k^2}{2}(\gamma-1)\right]t \end{pmatrix} + \begin{pmatrix} \tilde{A} & 0 & 0 \\ 0 & \tilde{B} & 0 \\ 0 & 0 & \tilde{C} \end{pmatrix} \right]$$

where $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ are constants of integration.

If we now define e^Q as the logarithm of a matrix J (see Coddi gton and Levinson page 65), i.e.,

$$e^Q = J = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}$$

and think of Q in the form

$$\begin{pmatrix} \log \mu_1 & 0 & 0 \\ 0 & \log \mu_2 & 0 \\ 0 & 0 & \log \mu_3 \end{pmatrix} .$$

Therefore, we may take the matrix J in the form

$$J = e^{Q} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e^{-iak\,Re\,t} & e^{iak\,Re\,t} \\ 0 & 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} e^{-\beta k^2 t} & 0 & 0 \\ 0 & e^{\left[-\frac{2}{3}k^2 - \frac{\beta k^2}{2}(\gamma-1)\right]t} & 0 \\ 0 & 0 & e^{\left[-\frac{2}{3}k^2 - \frac{\beta k^2}{2}(\gamma-1)\right]t} \end{pmatrix}$$

$$+ \begin{pmatrix} e^{A} & 0 & 0 \\ 0 & e^{B} & 0 \\ 0 & 0 & e^{C} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-\beta k^2 t} & 0 & 0 \\ 0 & e^{(-iak\,Re + M)t} & 0 \\ 0 & 0 & e^{(iak\,Re + M)t} \end{pmatrix} \begin{pmatrix} \tilde{A} \\ \tilde{B} \\ \tilde{C} \end{pmatrix}$$

where $M = -\frac{2}{3}k^2 - \frac{\beta k^2}{2}(\gamma-1)$ and $\tilde{A} = e^{A}$, $\tilde{B} = e^{B}$, $\tilde{C} = e^{C}$.
Thus, if we keep only the first term in the solution

$$\underline{x} = B_o\, e^{Re\,Q_o + Q_1}$$

we find that

$$\begin{pmatrix} dZ_k \\ dS \\ dQ \end{pmatrix} = \begin{pmatrix} 0 & ak & -ak \\ -1 & k & k \\ 0 & k(\gamma-1) & k(\gamma-1) \end{pmatrix} \begin{pmatrix} e^{-\beta k^2 t} & 0 & 0 \\ 0 & e^{(-iak\,Re+M)t} & 0 \\ 0 & 0 & e^{+(iak\,Re+M)t} \end{pmatrix}$$

$$\begin{pmatrix} \tilde{A} \\ \tilde{B} \\ \tilde{C} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & a k e^{(-iakRe+M)t} & -ak e^{(iakRe+M)t} \\ -e^{-\beta k^2 t} & k e^{(-iakRe+M)t} & k e^{(iakRe+M)t} \\ e^{-\beta k^2 t} & k(\gamma-1) e^{(-iakRe+M)t} & k(\gamma-1) e^{(iakRe+M)t} \end{pmatrix} \begin{pmatrix} A \\ \tilde{B} \\ \tilde{C} \end{pmatrix} .$$

$$dZ_k = ak e^{\left[-iakRe - \frac{2}{3}k^2 - \frac{\beta k^2}{2}(\gamma-1)\right]t} \tilde{B} - ak e^{\left[+iakRe - \frac{2}{3}k^2 - \frac{\beta k^2}{2}(\gamma-1)\right]t} \tilde{C} . \tag{A-3}$$

$$dS = -e^{-\beta k^2 t} A + k e^{\left[-iakRe - \frac{2}{3}k^2 - \frac{\beta k^2}{2}(\gamma-1)\right]t} \tilde{B} + k e^{\left[iakRe - \frac{2}{3}k^2 - \frac{\beta k^2}{2}(\gamma-1)\right]t} \tilde{C} . \tag{A-4}$$

$$dQ = e^{-\beta k^2 t} A + k(\gamma-1) e^{\left[-iakRe - \frac{2}{3}k^2 - \frac{\beta k^2}{2}(\gamma-1)\right]t} \tilde{B} + k(\gamma-1) e^{\left[iakRe - \frac{2}{3}k^2 - \frac{\beta k^2}{2}(\gamma-1)\right]t} \tilde{C} . \tag{A-5}$$

Let $t = 0$

$$dZ_k \big|_o = ak(\tilde{B} - \tilde{C}) \tag{A-6}$$

$$dS \big|_o = -A + k(\tilde{B} + \tilde{C}) \tag{A-7}$$

$$dQ \big|_o = A + k(\gamma-1)(\tilde{B} + \tilde{C}) . \tag{A-8}$$

Add Equations A-7 and A-8

$$dP \big|_o = \gamma k (\tilde{B} + \tilde{C})$$

and with

$$dZ_k \big|_o = ak(\tilde{B} - \tilde{C}) .$$

This yields for $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$

$$dP\big|_o/\gamma k + dZ_k\big|_o/ak = 2\tilde{B}$$

$$dP\big|_o/\gamma k - dZ_k\big|_o/ak = 2\tilde{C}$$

$$\tilde{A} = dQ\big|_o - (\gamma-1)\, dP_o/\gamma \quad .$$

Let $\alpha = \left[\frac{2}{3} + \frac{\beta}{2}(\gamma-1)\right]$.

Substituting back into the set of Equations A-1 through A-3,

$$dZ_k = dZ_k\big|_o \, e^{(-iakRe - \alpha k^2)t}$$

$$dS = \left(-dQ_o + \frac{(\gamma-1)}{\gamma} dP_o\right) e^{-\beta k^2 t} + \left(\frac{dZ_k\big|_o}{2a} + \frac{dP_o}{2\gamma}\right) e^{(-iakRe - \alpha k^2)t} + \left(\frac{dP_o}{2\gamma} - \frac{dZ_k\big|_o}{2a}\right) e^{(+iakRe - \alpha k^2)t}$$

$$dQ = \left(dQ_o - \frac{\gamma-1}{\gamma} dP_o\right) e^{-\beta k^2 t} + (\gamma-1)\left(\frac{dZ_k\big|_o}{2a} + \frac{dP_o}{2\gamma}\right) e^{(-iakRe - \alpha k^2)t} + (\gamma-1)\left(\frac{dP_o}{2\gamma} - \frac{dZ_k\big|_o}{2a}\right) e^{(+iakRe - \alpha k^2)t} \quad .$$

APPENDIX B

REVIEW OF THE LITERATURE

The mathematical theory of turbulence has a long history. Except in the simplest circumstances detailed understanding is far from complete. So far, most success which correlates with experimental results has been achieved for the special case of homogeneous isotropic incompressible turbulence as it is, for example, generated far down stream from wire grid in a wind tunnel. The theory for this case is based on the appropriate Navier-Stokes equations from which various correlation functions are constructed by taking averages on the equations themselves rather than on their solutions which are unknown. This process, unfortunately, always leads to an open set of moment equations because of the presence of the non-linear terms. Thus, with this approach some additional hypothesis is necessary in order to close the system. In the simplest case this merely consists in neglecting all correlation functions of higher than the second order; but a great many more subtle approaches have also been used, including the quite widely used quasi-Gaussian approximation which relates fourth order correlation functions to those of second order. The use of spectral analysis via methods of Fourier transformation have also been extensively exploited and much work has been devoted to the hypotheses introduced by Heisenberg and Kolmogoroff (the so-called locally homogeneous and locally isotropic turbulence), whose purpose was to determine the decay laws. We will not enter into any of these matters here in detail for they are nicely described in the book by G. K. Batchelor (The Theory of Homogeneous Turbulence, Cambridge Univ. Press, 1953) as well as in the book by L. Agostini and J. Bass (Les Theories de la Turbulence, Publ. Sci. Tech. Ministere L'air, no. 237, 1950).

In addition to the approach described above there are at least two other distinct methods of attack which are conceptually more appealing. The first consists in the deduction of a functional equation, equivalent to an infinite set of equations, whose solution where it is known would yield exact expressions for the various correlation functions of interest. This approach was initiated by E. Hopf (Statistical Hydromechanics and Functional Calculus, J. Ratl. Mech. Anal., vol 1, 1952, pp 87-123)

in a space formalism and recently extended to a space time formalism by R. Lewis and R. Kraichnan (A Space-Time Functional Formalism for Turbulence, Comm. Pure and App. Math. 15, 1963, p 363).

The second method involves the representation of the various quantities such as velocities, density, temperature, etc., in terms of Fourier transforms of singular random functions or fields. The basic concepts are briefly discussed in Batchelor's book on pages 28-33 and in the basic paper by Moyal on which most of this report is based (The Spectra of Turbulence in a Compressible Fluid; Eddy Turbulence, and Random Noise, Proc. Camb. Phil. Soc. 48, 1952, page 329). The relevant mathematical background is discussed by Moyal (Stochastic Processes and Statistical Physics, J. Royal Statistical Soc., B, 11, 1949, p 150). This approach has also been the subject of much Russian work as may be seen by reference to the article by A. Yaglom (Some Classes of Random Field in n-Dimensional Space, Related to Stationary Random Processes, Theory of Probability and its Applications, 2, no. 3, 1957, p 273). While this work too has largely been confined to characterizations involving incompressible flows, the paper by Moyal is one of the few that we have discovered which lays a foundation on which one could base a long-range systematic attack on the problems of interest. In fact, our search of the literature* for material relating to compressible turbulent effects proved most disappointing, although not unexpectedly so, because of the intrinsic difficulty of the problem. In particular, Moyal achieves a partial separation in wave vector space of the eddy forming part of the velocity (the transverse part of the singular random field) from the remainder (the longitudinal part) which allows a somewhat easier physical interpretation of the involved complicated phenomena. Before entering into Moyal's results, it is perhaps worthwhile to review in brief some of the facts concerning the nature of turbulence as expressed by three distinguished mathematicians: J. Leray, E. Hopf, and J. von Neumann.

We begin this review by quoting in free translation some excerpts from Leray's paper (Sur le Mouvement d'un Liquide Visqueux Emplissant L'espace, Acta. Math., 63, 1934, p 193).

*A rather thorough and systematic literature search was performed using Mathematical Reviews, Science Abstracts, and DDC.

"The theory of viscosity of liquids is governed by the equations of Navier. It is necessary to justify a posteriori this hypothesis by establishing the existence theorem which follows. There exists a solution of the equations of Navier which corresponds to a given arbitrary initial set of values of the velocities. Oseen tried to prove this—he only succeeded in establishing the existence of such a solution for a period of time, perhaps very short, succeeding the initial instant. Now one can verify that the total kinetic energy of the fluid remains bounded; but it does not seem possible to deduce from this the fact that the flow remains regular . . . .

"It is not paradoxial to suppose that the cause regulating the flow—the dissipation of energy—is not sufficient to keep the second derivatives of the velocity components bounded and continuous with respect to their coordinates; now the theory of Navier assumes that the second derivatives are bounded and oontinuous. Oseen himself commented on the unnatural character of this hypothesis; at the same time he showed how to express the laws of fluid mechanics in terms of integrodifferential equations in which only the components of velocity and their first derivatives with respect to the coordinates occur. In the course of this work I will consider a system of relations which is equivalent to the integrodifferential equation of Oseen, completed by an inequality expressing the dissipation of energy. These relations are deduced from the equations of Navier by the aid of integration by parts which makes the highest order derivatives disappear. (Today these are the so-called weak solutions.)* And if I have not succeeded in establishing the existence theorem stated above, I have nevertheless proved the following: the relations in question (i. e., the weak form) always possess at least one solution which is defined for an unlimited time and which corresponds to a given initial state of velocity. Perhaps this solution is not regular enough to possess second derivatives which are bounded at each instant of time; nonetheless it is, in a proper sense of the term, a solution of the equations of Navier; I propose to call them turbulent solutions.

"If I had succeeded in constructing the solutions of the Navier equations which became irregular, I would then

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*Authors' note.

be able to affirm that turbulent solutions exist which are not at all simply reducible to regular solutions. But if this is false, the concept of a turbulent solution will not lose its interest since it is bound to play a large role in the study of viscous liquids; it is well designed to handle the problem of physics and mathematics where the physical causes of regularity are not sufficient to include the hypotheses of regularity which are consequences of the deduced equation; one can apply the considerations presented here to these problems.

"We note the following two facts:

"Nothing permits one to assert the uniqueness of the turbulent solution which corresponds to a given initial state;

'The solution which corresponds to a state sufficiently near to a steady state can never become irregular."

To express the philosophy of E. Hopf we quote from A Mathematical Example Displaying Features of Turbulence, Comm. Pure and App. Math., 1, 1948, p 303.

"It is convenient to visualize the solutions in the phase space of the problem. A phase or state of the fluid is a vector field . . . in the fluid space that satisfies the (Navier-Stokes) equations and the boundary conditions. The totality Ω of these phases is therefore a functional space with infinitely many dimensions. A flow of the fluid represents a point motion in Ω and the totality of these phase motions form a stationary flow in the phase space Ω, which, of course, is to be distinguished from the fluid flow itself. What is the asymptotic future behavior of the solutions, how does the flow behave for $t \to \infty$, and how does this behavior change as ν (the viscosity) decreases more and more? How do the solutions which represent the observed turbulent motions fit into the phase picture? . . .

"The observational facts about hydrodynamic flow reduced to the case of fixed side conditions and with ν as the only variable parameter are essentially these; for ν sufficiently large, $\nu > \nu_0$, the only flow observed in the long run is a stationary one (Laminar flow). This flow is stable against arbitrary initial disturbances. Theoretically the corresponding exact solution is known to exist for every value of $\nu > 0$

and its stability in the large can be rigorously proved, though only for sufficient large values of ν. The corresponding flow in phase space Ω thus possesses an extremely simple structure. The laminar solution represents a single point in Ω invariant under the phase flow. For $\nu > \nu_o$ every phase motion tends as $t \to \infty$ toward this laminar point. For sufficiently small values of ν, however, the laminar solution is never observed. The turbulent flow observed instead displays a complicated pattern of apparently irregular moving 'eddies' of varying sizes. The view widely held at present is that for $\nu > 0$, having a fixed value, there is a 'smallest size' of eddies present in the fluid tending to zero as $\nu \to 0$. Thus, macroscopically, the flow has the appearance of an intricate chance movement whereas, if observed with sufficient magnifying power, the regularity of the flow would never be doubted.

"The qualitative mathematical picture which the author conjectures to correspond to the known facts about hydrodynamic flow is this; to the flows observed in the long run after the influence of the initial conditions has died down there corresponds certain solutions of the Navier-Stokes equations. These solutions constitute a certain manifold $M = M(\nu)$ in phase space invariant under the phase flow. Presumably owing to viscosity M has a finite number $N = N(\nu)$ of dimensions. This effect of viscosity is most evident in the simplest case of ν sufficiently large. In this M is simply a single point $N = 0$. Also the complete stability of M is, in this simplest case, obviously due to viscosity. On the other hand, for smaller and smaller values of ν, the increasing change characteristic of the observed flow suggests that $N(\nu) \to \infty$ monotonically as $\nu \to 0$. This can happen only if at certain 'critical' values

$$\nu_o > \nu_1 > \nu_2 \ . \ . \ . \to 0$$

the number $N(\nu)$ jumps. The manifold $M(\nu)$ itself presumably changes analytically as long as no critical value is passed. Now we believe that when ν decreases through such a value ν_K, a continuous branching phenomenon occurs. The manifold $M(\nu)$ of motions observed in the long run (more precisely its analytical continuation for $\nu < \nu_K$) loses its stability.

The notion of stability here refers to the whole manifold and not to the single motions contained in it. The loss of stability implies that the motions on the analytically continued M are no longer observed. What we observe after passing ν_K is not the analytical continuation of the previous M but a new manifold $M(\nu)$ continuously branching away from $M(\nu_K)$ and slightly swelling in a new dimension. This new $M(\nu_K)$ takes over stability from the old one. Stability here means that the 'majority' of phase motions tend for $t \to \infty$ toward $M(\nu)$. We must expect there is a 'minority' of exceptional motions that do not converge toward M (for instance the motions on the analytical continuation of the old M and of all the other manifolds left from all the previous branchings). The simplest case of such a bifurcation with corresponding change of stability is the branching of a periodic motion from a stationary one. This case is clearly observed in the flow around an obstacle (transition from the laminar flow to a periodic one with periodic discharge of eddies from the boundary). The next simplest case is the branching of a one-parameter family of almost periodic solutions from a periodic one."

In this same paper, Hopf gave a simple example of a mathematical model which displayed the features described above. Later in a paper entitled, On Non-Linear Partial Differential Equations, which appeared in the Lecture Series of the Symposium on Partial Differential Equations, published by the University of Kansas in 1957, he gave a still simpler example. However, suggestive as these examples are, in no sense are they complicated enough to lead to any real insight into the nature of actual turbulent flow nor do they suggest any attack on the main problem even for the incompressible case.

Finally, we will quote a few excerpts from a privately circulated report by the late John von Neumann, dated 12 June 1950. This report was made available to us through the courtesy of Professor G. Uhlenbeck of the Rockefeller Institute.

"It is, by the way, obvious that one has to talk not of one turbulent solution, but of many turbulent solutions. The problems which lead to turbulence are usually such that the outer conditions of the problem do not change with time, whereas the turbulent solution itself invariably does. A translation of the whole solution along the time-axis is

therefore always possible and produces infinitely many solutions of the same problem, all possessing the same turbulent characteristics as the original solution. Indeed, there is every reason to believe that in addition to this, many other principles of transforming these solutions exist. There is probably no such thing as a most favored or most relevant, turbulent solution. Instead, the turbulent solutions represent an ensemble of statistical properties, which they all share, and which alone constitute the essential and physically reproducible traits of turbulence.

"In the classical setup, statistical mechanisms, the system under consideration is isolated from the outside world. Its energy is fixed, and one is concerned with the to-and-fro energy exchanges between degrees of freedom. The classical theory allows to infer from this that equipartition of energy between all degrees of freedom prevails. In the present setup, on the other hand, the system is 'open' at both ends, energy is being supplied as well as dissipated. The two 'ends' do not, however, lie in ordinary space, but in its Fourier transform. More specifically: The supply of energy occurs at the macroscopic end—it originates in the forced motions of macroscopic (bounding) bodies, or in the forced maintenance of (again macroscopic) pressure gradients. The dissipation, on the other hand, occurs mainly at the microscopic end, since it is ultimately due to molecular friction, and this is most effective in flow-patterns with high velocity gradients, that is, in small eddies. (While the main dissipative flow-patterns are microscopic, they are not necessarily molecular, ...) That is: A flow of energy is taking place from the sources which are situated at the low frequencies to the sinks that lie in the high frequencies. (The frequencies are to be understood in space, and not in time.) Thus, the statistical aspect of turbulence is essentially that of a transport phenomenon (of energy)—transport in the Fourier-transform space.

"Having understood this decisive trait—that turbulence is not a matter of ergodic distribution of a fixed amount of energy, but the transport of a fixed flow of energy from sources in the low frequencies to sinks in the high frequencies in the Fourier-transform space—it is now possible to take the next step.

"This step consists of noting that generally valid and simple phenomena should be expected in the domain of pure flow (energy flow, in the Fourier-transform space), remote from both the sources and the sinks, that is, in the intermediate frequencies. Indeed, the sources, as well as the sinks cause specific complications, depending on more complicating factors with which it is advisable to deal separately and subsequently. A simple and unique phenomenon, which is of the same nature under all possible conditions of turbulence, is to be expected in the domain of pure energy flow only, as indicated above.

"This circumstance was recognized and primarily dealt with in the work of group C.* The discussion which follows will accordingly begin with emphasizing this aspect. The variant to be presented is a composite one, using elements of the approaches of several authors.

"The intermediate (spatial) frequency range corresponds clearly to an intermediate size range, that is, to an intermediate vorticity-and-eddy-size range. Note again: This range has to be defined as being too small to be affected by the bounding solids (energy sources), but at the same time too large to be affected by high- (molecularly viscous) dissipation eddies (energy sinks). Hence there should be no influence of linear dimensions (of the macroscopic system): L , and also no influence of the (molecular) viscosity: ν . Note further: Yet, these enter into the Reynolds number $R = \frac{LU}{\nu}$, which was heretofore viewed as the decisive quantity: Thus the only quantity that can matter is the rate of dissipation W (= energy/unit mass × unit time).

"The following circumstance is also worth noting: It need not be true that the mere cessation of stability of the laminar flow is already equivalent to the onset of a properly turbulent regime. It is quite normal to expect that for Reynolds numbers only slightly in excess of the stability limit only one unstable mode of perturbation exists. The non-laminar form of motion which then develops, will therefore still be one of relative simplicity and not at all resembling the familiar, highly involved, pattern of turbulence.

*Author's note: The work of A. N. Kolmogoroff, L. Onsager, C. F. von Weizsäcker and W. Heisenberg.

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(All available conclusive results in laminar-perturbation stability theory confirm this . . . .) The development of such an unstable mode of perturbation, that is, of one which increases with time, nevertheless may start a chain of events which ultimately leads to turbulence. The plausible picture of this evolution may be somewhat like this: As long as the (unstable) perturbing mode has not developed to a large size, the stability properties of the perturbed (that is, distorted) laminar flow will be essentially the same ones as those of the unperturbed (original) one. That is, no new unstable modes come into play. If the perturbing mode reaches, however, a certain finite size—at which time, of course, the applicability of linear theory will have ceased and a non-linear discussion will have become necessary—the stability properties may shift. The now significantly perturbed (distorted) flow may now acquire further unstable modes which will also begin to grow. They are likely to catch up with the primary unstable mode in size and importance, because the primary mode is already in the non-linear range and therefore probably no longer increasing, while the secondary modes will be exponentially increasing as long as they are small, that is, until they themselves acquire a finite size and leave the linear range. As long as these secondary modes are small, they will cause no change of the stability properties. When they become large, however, they may in their turn modify the stability situation and give rise to ternary unstable modes, etc. It is clear how this mechanism successively produces the degree of involvement which characterizes turbulence in the ordinary sense. It is plausible to say that turbulence proper has set in when each new family of increasing unstable modes is able, after having reached a certain finite size, to cause the generation of a subsequent family of unstable modes, etc. The first ones in this chain of consecutive unstable perturbations may be said to belong to the low frequency range, the subsequent ones, which generate each other by an inductive process, form the intermediate frequency range.

"Consider a fluid, which will, for the sake of simplicity, be considered as (strictly) non-viscous and (strictly) infinitely (electrically) conductive. If an electromagnetic field is present, then the magnetic lines of force are, as

is well known, 'frozen' to the fluid—that is, for all motions of the fluid each line of force will move strictly with the material particles with which it coincided originally. If the fluid executes a complicated motion, then these lines of force will assume very involved shapes, they will therefore produce large gradients, that is, large magnetic forces, and hence large electromagnetic energies. If the fluid is now allowed a small viscosity, and the electromagnetic field is at first weak, then the fluid's motion will be essentially turbulent, and the electromagnetic energies will be greatly increased. This mechanism will continue to play, as long as the electromagnetic field is small. Thus it is an amplifying mechanism, which continues to operate until it has amplified itself out of the linear (the amplifying) range—that is, until the electromagnetic phenomena have risen to the same order of importance as the purely hydrodynamical ones.

"Let the fluid now possess a large, but not infinite, conductivity (i. e., a small, but not zero, resistivity). Then the phenomena will occur essentially as described above, except that the magnetic lines of force do now possess a small mobility with respect to matter. This mobility, no matter how small, will certainly become important in the regions of sufficiently high involvement of the magnetic lines of force—that is, where they have become sufficiently tortuous, and sufficiently closely pressed together. Here the finite conductivity (i. e., the non-zero resistivity) will cause an energy dissipation.

"Note the similarity between this mechanism and that one of dissipation by viscosity: Any non-zero viscosity coefficient, no matter how small, must be able to take care of any given turbulent dissipation of energy, by failing to inhibit the formation of sufficiently small eddies, and then working on the correspondingly high velocity gradients which are present in these. Similarly, any finite conductivity, no matter how large (i. e., any non-zero resistivity, no matter how small), may be able to take care of a given dissipation of energy, by failing to inhibit the formation of sufficiently involved and sufficiently closely compressed magnetic lines of force, and then working on the high fields created by these and by their motion. Thus there are

'electromagnetic eddies very much like the purely hydrodynamical ones, and conductivity (or rather its reciprocal, resistivity) may be able to bear to turbulence very near the same relationship as viscosity.

"From the point of view of theoretical physics, turbulence is the first clearcut instance calling for a new form of statistical mechanics: As the earlier discussions have shown, it requires the establishing of general statistical laws for energy transport phenomena in the Fourier-transform space, that is, of the adjustment of the degrees of freedom of a system, which is endowed with many degrees of freedom, to a flow of energy which enters it in one set of frequencies and leaves it in another. (. . . .) The existing theories (especially that one of Kolmogroff Onsager and Heisenberg) suffice to show that these laws will differ essentially from those of classical (Maxwell-Boltzmann-Gibbsian) statistical mechanics. Thus it is certain that the law of equipartition of energy between all degrees of freedom, which is valid in the latter, is replaced by something altogether different in the former."

In addition to the standard references listed above, we have relied upon Turbulent Flows and Heat Transfer, editor: C. C. Lin, Princeton University Press, for the literature up to 1959. The pertinent sections of this book are Chapters 2 and 5 of Part B and Chapters 1 through 5 of Part C. For references up to 1962, we have relied heavily on International Symposium on Fundamental Problems in Turbulence and their Relation to Geophysics, Journal of Geophysical Research, vol 67, no. 8, July 1962. Of all the papers presented in this symposium titles and abstracts of those most pertinent are the following:

1. Turbulence in the Presence of a Vertical Body Force and Temperature Gradient, by Robert G. Deissler.

Two-point correlation equations which include the effects of a uniform temperature gradient and body force are constructed from the Navier-Stokes, heat-transfer, and continuity equations. A solution is obtained by converting the correlation equations to spectral form and assuming that the turbulence is sufficiently weak for triple correlations to be neglected. It is shown that the turbulence decays with time, although the rate of decay is altered by buoyance effects caused by the body force and temperature gradient. The buoyancy forces can either extract energy from the turbulent field or feed into it, depending

on the directions of the body force and temperature gradient. Spectra are calculated for the turbulent energy and for the various terms in the turbulent energy equation as well as for the temperature fluctuations and turbulent heat transfer. For fluids with Prandtl numbers less than 1 the buoyancy forces act mainly on the large eddies, whereas for higher Prandtl numbers they can act on the smaller ones. When the buoyancy forces are stabilizing, the turbulence can cause heat to flow against the temperature gradient for certain values of the parameters. For making the calculations, it is assumed that the turbulence is initially isotropic and the temperature fluctuations initially zero.

2. Some Mathematical Models Generalizing the Model of Homogeneous and Isotropic Turbulence, by A. M. Yaglom.

Homogeneous and isotropic turbulence is an example of a system of random fields invariant with respect to a group of motions. Along with homogeneous and isotropic fields, locally homogeneous and locally isotropic ones play an important role in turbulence theory; such local fields may also have an accurate mathematical definition. The random fields invariant with respect to groups of transformations different from a group of Euclidean motions can also be considered; the 'spectral representation' of such a field and of a corresponding correlation function often has an unusual form, although its sense remains the same. The algebraic theory of group representations gives the general method of obtaining the spectral representation for the fields. The examples of homogeneous random fields on a sphere and fields in a semiplane invariant with respect to all similarity transformations present interesting examples of random fields invariant with respect to 'motions' of special type which might be of some importance for turbulence theory.

Some other related papers by Yaglom follow:

Homogeneous and Isotropic Turbulence in a Viscous Compressible Fluid, Izvest. Akad. Nauk SSSR, Ser. Geograph. and Geophys., 12(6), 1948; see also German translation of this paper in Statistische Theorie der Turbulenz, Akademie-Verlag, Berlin, 1958.

Some Classes of Random Fields in n-Dimensional Space Related to Stationary Random Processes, Theory of Probability and Its Application, 2(3), 1957.

Positive Definite Functions and Homogeneous Random Fields on Groups and Homogeneous Spaces, Doklady Akad. Nauk SSSR, 135(6), 1960.

Second-Order Homogeneous Random Fields, Proc. 4th Berkeley Symposium on Math. Statistics and Probability, 2, Berkeley-Los Angeles, 1961.

3. Energy Transfer in an Isotropic Turbulent Flow, by Yoshimitsu Ogura.

This paper examines the dynamic consequence of the hypothesis that fourth-order mean values of the fluctuating velocity components are related to second-order mean values as they would be for a normal joint-probability distribution. The equations derived by Tatsumi for isotropic turbulence on the basis of this hypothesis are integrated numerically as an initial value problem for an inviscid fluid. The most remarkable feature revealed by the computation is that the energy spectrum function becomes negative during the course of time in certain regions of wave-number space. This situation is similar to the result obtained previously for two-dimensional turbulence. Truncation errors that arise from finite-difference approximations in numerical integration are examined. It is tentatively concluded that this unphysical negative energy is not generated by the truncation errors but is the consequence of the quasi-normality hypothesis.

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Tatsumi, T., The Theory of Decay Process of Incompressible, Isotropic Turbulence, Proc. Roy. Soc. London, A, 239, 16-45, 1957.

4. Perturbation Analysis of the Navier-Stokes Equations in Lagrangian Form with Selected Linear Solutions, by Willard J. Pierson.

The Navier-Stokes equations for incompressible flow in their Lagrangian form are taken as a starting point. A perturbation technique is then used to obtain first- and second-order sets of equations, and the general procedure for solving the equations to any order is given. The first-order equations yield interesting two- and three-dimensional motions that have some of the properties of 'stirring,' 'eddies,' and 'turbulence;' it is suggested that various problems in turbulent motion might possibly be re-examined by means of these equations.

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In addition, we have made a systematic search of Mathematical Reviews, Science Abstracts, and the DDC abstracts. The papers bearing on the problem at hand with abstracts when available are listed below.

Elliott, L. A., Similarity Methods in Radiation Hydrodynamics, Proc. Roy. Soc., vol 258, A. (25 October 1960).

The extent to which problems in radiation hydrodynamics may be tackled by similarity methods is investigated for the case of spherical symmetry. In particular the problem of an intense explosion is considered; by the introduction of radiative heat flux the singularity in a solution by Taylor (1950) is removed.

Chandrasekhar, S., On Turbulence Caused by Thermal Instability, Proc. Roy. Soc., vol 244, A. 884 (11 March 1952).

In this paper a statistical theory of turbulence in incompressible fluid caused by the joint effects of gravity, and thermal instability, is developed. The mathematical theory is based on the equations of continuity and heat conduction and the Boussinesq form of the equations of motion in which the variations of density (resulting from the variations in temperature) are taken into account only in so far as they modify the action of gravity. By restricting oneself to a portion of the fluid far from the bounding surfaces one can treat the turbulence as approximately homogeneous and axisymmetric

and use the theory of axisymmetric vectors and tensors recently developed by the author (Chandrasekhar 1950a). A number of correlations between the various field quantities (such as the velocity components, fluctuations in temperature, etc.) at two different points in the medium are defined; and a closed system of equations for the defining scalars are derived for the case when the non-linear terms in the equations of motion and heat conduction can be neglected and a constant mean adverse temperature gradient is maintained. Under stationary conditions when the time derivatives of the various correlations are zero, there is an exact balance between the dissipation of kinetic energy by viscosity and the liberation of potential energy by gravity.

A fundamental set of solutions of the equations governing stationary turbulence is obtained; these solutions, varying periodically in the vertical direction, enable a generalized Fourier analysis of the various correlation functions. According to these solutions, a Fourier analysis of correlations such as $\overline{u_{11}(0)\, u'_{11}(z)}$ of the vertical velocities at two points directly above one another and separated by a distance z, cannot include wavelengths less than a certain minimum value depending on the physical parameters and on the temperature gradient maintained. We may thus speak of a smallest size for the eddies. Further, it appears that the field of turbulence can be analyzed into two modes characterized by the kinetic energy being confined, principally, to the vertical or to the horizontal direction.

Moore, D. W., The Boundary Layer on a Spherical Gas Bubble, Journal of Fluid Mech., vol 16.

The equations governing the boundary layer on a spherical gas bubble rising steadily through liquid of small viscosity are derived. These equations are linear and are solved in closed form. The boundary layer separates at the rear stagnation point of the bubble to form a thin wake, whose structure is determined. Thus the drag force can be calculated from the momentum defect. The value obtained is $12\pi a U \mu$, where a is the bubble radius and U the terminal velocity, and this agrees with the result of Levich (1949) who argued from the viscous dissipation in the potential flow round the bubble. The next term in an expansion of the drag in descending fractional powers of R is found and the results compared with experiment.

Pao, Yih-Ho, Growth of a Weak Magnetic Field in a Turbulent Conducting Fluid with Large Magnetic Prandtl Number, The Physics of Fluids, vol 6, no. 5.

The behavior of a weak fluctuating magnetic field in a turbulent electrically conducting fluid with a large magnetic Prandtl number is reexamined, without using vorticity analogy. For a large magnetic Prandtl number, the characteristic length of the small eddies of turbulence is much larger than that of the small loops of the magnetic field. By inferring that the small eddies of turbulence are mainly responsible for the change of the fluctuation intensity of the magnetic field and by utilizing Batchelor's mixing approach, visualizing the turbulent velocity field as effectively a rather persistent uniform straining motion for the small-scale variation of magnetic field, it was found that the fluctuation intensity of the magnetic field grows due to turbulent motion for very large magnetic Prandtl numbers [$(\nu/\lambda) > 100$, say] and approaches a constant value asymptotically. The rate of growth is found to be a function of ν/λ and τ (the product of time and the turbulent straining rate).

Chandrasekhar, S., The Onset of Convection by Thermal Instability in Spherical Shells, Philosophy Magazine, ser 7, vol 44, no. 350, March 1953.

In this paper the problem of the thermal instability of an incompressible sphere consisting of an inviscid core and a viscous mantle is considered, and it is shown that the pattern of convection which sets in, at marginal stability, in the mantle shifts to harmonics of the higher orders as the thickness of the mantle decreases. Thus, when the mantle extends to a depth of half the radius of the sphere, the harmonics of orders three and four set in about simultaneously, while the harmonic of order five follows very soon afterwards. The bearing of this result on the problem of convection in the earth's mantle and of the interpretation of the earth's topographic features is indicated.

Chandrasekhar, S., The Decay of Axisymmetric Turbulence, Proc. Roy Soc., vol 203 (10 October 1950).

In this paper the decay of axisymmetric turbulence is investigated. Explicit solutions, appropriate for the final period of decay, are obtained; these solutions are in agreement with Batchelor's general results on the role of the big eddies in homogeneous turbulence. It is further shown that, compatible with the equations which have been derived for axisymmetric turbulence, a state of turbulence exists which may be pictured as a superposition of two non-interacting fields which are isotropic and axisymmetric, respectively.

Tatsumi, T., The Theory of Decay Process of Incompressible Isotropic Turbulence, Proc. Roy Soc. 239A, p 16.

One of the most serious difficulties in the theory of homogeneous turbulence is the indeterminancy of the equation for the velocity correlation function of any order, each involving the correlation of higher-by-one order. In the present paper this difficulty is resolved by treating the two dynamical equations for the second- and third-order velocity correlations, and by introducing the assumption of the zero fourth-order cumulant of the velocity field which yields a relationship between the fourth- and second-order velocity correlations. Actual calculation, however, is carried out in the wave-number space, and a pair of simultaneous equations for the energy spectrum function are derived in Part I.

Another difficulty of the subject arises from the present lack of knowledge about the initial state of turbulence. In Part II, some probable initial conditions for the energy spectrum are examined, among which the initial spectrum of single-line type is chosen as the most suitable for the present problem and its dynamical consequences are fully discussed. The power-series solution for the initial spectrum as well as the energy decay law due to it are computed and compared with experimental data. It is found that the solution, in so far as the approximate expression calculated in the present paper is concerned, corresponds to the earlier initial period of decay. A solution which would be essentially in agreement with experiments is expected to be given by extending the present solution to the further developed stage of decay.

Morton, B. R., Sir Geoffrey Taylor, and J. S. Turner, Turbulent Gravitational Convection from Maintained and Instantaneous Sources, Proc. Roy. Soc., vol 234, A. (24 January 1956).

Theories of convection from maintained and instantaneous sources of buoyancy are developed, using methods which are applicable to stratified body fluids with any variation of density with height; detailed solutions have been presented for the case of a stably stratified fluid with a linear density gradient. The three main assumptions involved are:

1. That the profiles of vertical velocity and buoyancy are similar at all heights

2. That the rate of entrainment of fluid at any height is proportional to a characteristic velocity at that height

3. That the fluids are incompressible and do not change volume on mixing, and that local variations in density throughout the motion are small compared to some reference density.

The governing equations are derived in non-dimensional form from the conditions of conservation of volume, momentum and buoyancy, and a numerical solution is obtained for the case of the maintained source. This leads to a prediction of the final height to which a plume of light fluid will rise in a stably stratified fluid. Estimates of the constant governing the rate of entrainment are made by comparing the theory with some previous results in uniform fluids, and with the results of new experiments carried out in a stratified salt solution.

For the case of an instantaneous source of buoyancy there is an exact solution; the entrainment constant is again estimated from laboratory results for a stratified fluid.

Finally, the analysis is applied to the (compressible) atmosphere, by making the customary substitution of potential temperature for temperature. Predictions are made of the height to which smoke plumes from typical sources of heat should rise in a still, stably stratified atmosphere under various conditions.

Chandrasekhar, S., Hydromagnetic Turbulence. I. A Deductive Theory.

In this paper a deductive theory of turbulence recently described by the writer (Chandrasekhar 1955a) is extended to hydromagnetics. By considering stationary turbulence and making statistical hypotheses of the same general character as in the hydrodynamical theory, a pair of differential equations are derived for the scalars defining the isotropic tensors describing the correlation in the velocities and the intensities of the magnetic field at two different points and at two different times.

Batchelor, G.K., The Theory of Axisymmetric Turbulence, Proc. Roy. Soc., vol 186. A.

This paper discusses a type of turbulence in a uniform stream which is next to isotropic turbulence in order of simplicity. Instead of spherical symmetry, or isotropy, axially symmetrical turbulence possesses symmetry about an axis which in practice is usually the direction of mean flow. The analysis is developed with the aid of invariant theory, as suggested by a previous paper by Robertson. The form of the fundamental velocity correlation is obtained, and scales of axisymmetric turbulence are defined.

The results of greatest practical interest concern the time rates of change of the mean squares of the lateral and longitudinal velocity

components. The rates of change involve two terms, the first representing viscous dissipation, and the second representing a transfer of energy from one component to the other due to the finite correlation between the velocity and pressure at neighbouring points. The effect of the velocity-pressure correlation is to bring the two velocity components towards equality, while the effect of the viscous dissipation will only be towards equality if an inequality between the curvatures at the origin of two particular velocity correlation coefficient curves, both of which are measurable, is obeyed. The rates of change of the mean squares of the vorticity components are also obtained.

Chandrasekhar, S., The Thermal Instability of a Fluid Sphere Heated Within, Philosophy Magazine, vol 43, December 1952.

In this paper the problem of the thermal instability of an incompressible fluid sphere heated within and in equilibrium under its own gravitation is considered. A general disturbance is analysed into modes in terms of spherical harmonics of various orders, l, and the criterion for the onset of convection for the first fifteen modes is found both when the bounding surface is free and when it is rigid; and it is shown that in both cases the mode l = 1 is the first to be excited.

Walters, J. K. and J. F. Davidson, The Initial Motion of a Gas Bubble Formed in an Inviscid Liquid, Part 1. The Two-Dimensional Bubble, J. of Fluid Mechanics, vol 12, no. 408, March 1952.

The paper deals with the initial motion of a two-dimensional bubble starting from rest in the form of a cylinder with its axis horizontal. The theory is based on the assumptions of irrotational motion in the liquid round the bubble, constant pressure within the bubble, and small displacements from the cylindrical form. This theory predicts that the bubble should rise with the acceleration of gravity, over a distance of at least the initial bubble radius, and that a tongue of liquid should be projected up from the base of the bubble into its interior. These predictions are confirmed by experiments which also show how the vorticity necessary for steady motion in the spherical-cap form is generated by the detachment of two small bubbles from the back of the main bubble.

APPENDIX C

EVALUATION OF INTEGRALS

The inversion in Equation 23 follows from

$$\mathrm{Inv}\left[e^{-k^2 t}/k^2\right] = \int_0^{2\pi}\int_0^{\infty}\int_0^{\pi} \frac{e^{-k^2 t + k r \cos\alpha}}{k^2} \, k^2 \sin\alpha \, d\alpha \, dk \, d\beta$$

$$= 4\pi \int_0^{\infty} e^{-k^2 t} \frac{\sin k r}{k r} \, dk$$

$$= \frac{-2\pi^2}{r} \operatorname{erf}\left(\frac{r}{2\sqrt{t}}\right)$$

The last integral may be evaluated by integrating with respect to r both sides of the following relation

$$\int e^{-k^2 t} \cos k r \, dk = \frac{\sqrt{\pi}}{2\sqrt{t}} e^{-r^2/4t}$$

this gives

$$-\int e^{-k^2 t} \frac{\sin k r}{k} \, dk = \frac{\pi}{2} \operatorname{erf}\left(\frac{r}{2\sqrt{t}}\right)$$

The quantity $\operatorname{erf}\left(\frac{r}{2\sqrt{t}}\right)$ is the error function given by

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^{x} e^{-u^2} \, du$$

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13. ABSTRACT This report describes a technique for the solution of a class of problems in compressible turbulent flow and demonstrates its application to a problem involving the rise and the entrainment of ambient air by the late-time fireball due to a nuclear detonation in the atmosphere. The method is based on the work of J. E. Moyal which shows that the components of Fourier spectra of velocity parallel to the wave vector are associated with compression waves (irrotational flow) while those components transverse to the wave vector are associated with eddy turbulence. If solutions are sought for systems in pressure equilibrium with an undisturbed atmosphere, the resulting formalism takes on considerable simplicity due to uncoupling of the equations of motion. (U)

Assumptions leading to the evaluation of certain convolution integrals which appear in the theory are examined and shown to involve the largest eddies in the role of the principal convective agent. This assumption converts the nonlinear Navier-Stokes equations to linear form with solutions conveniently described in a moving frame of reference. The heat transfer and continuity equations are also simplified by the same linearization process. Simultaneous solutions of the equations in wave-vector space and subsequent inversion to physical space with certain initial conditions provides solutions which have considerable similarity to the actual flow situation found in weapons tests such as vortex ring (torus) formation. Qualitative and quantitiative features of the predicted motion, for example, velocity distribution, rise rate, and temperature are discussed. (U)

A section on conclusions and recommendations assembles the main ideas and results which have come from the study and suggests the direction of future effort both with respect to the fireball problem and with respect to the general theory of turbulent flow. (U)

14. KEY WORDS	LINK A ROLE	LINK A WT	LINK B ROLE	LINK B WT	LINK C ROLE	LINK C WT
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Late-Time Fireball						
Gravitational Convection						
Turbulence						
Navier-Stokes Equation						
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Linearization						
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